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# CHARACTERISATION OF A FAMILY OF NEIGHBOUR TRANSITIVE CODES

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**ABSTRACT.** We consider codes of length  $m$  over an alphabet of size  $q$  as subsets of the vertex set of the Hamming graph  $\Gamma = H(m, q)$ . A code for which there exists an automorphism group  $X \leq \text{Aut}(\Gamma)$  that acts transitively on the code and on its set of neighbours is said to be *neighbour transitive*, and were introduced by the authors as a group theoretic analogue to the assumption that single errors are equally likely over a noisy channel. Examples of neighbour transitive codes include the Hamming codes, various Golay codes, certain Hadamard codes, the Nordstrom Robinson codes, certain permutation codes and frequency permutation arrays, which have connections with powerline communication, and also completely transitive codes, a subfamily of completely regular codes, which themselves have attracted a lot of interest. It is known that for any neighbour transitive code with minimum distance at least 3 there exists a subgroup of  $X$  that has a 2-transitive action on the alphabet over which the code is defined. Therefore, by Burnside's theorem, this action is of almost simple or affine type. If the action is of almost simple type, we say the code is *alphabet almost simple neighbour transitive*. In this paper we characterise a family of neighbour transitive codes, in particular, the alphabet almost simple neighbour transitive codes with minimum distance at least 3, and for which the group  $X$  has a non-trivial intersection with the base group of  $\text{Aut}(\Gamma)$ . If  $C$  is such a code, we show that, up to equivalence, there exists a subcode  $\Delta$  that can be completely described, and that either  $C = \Delta$ , or  $\Delta$  is a neighbour transitive frequency permutation array and  $C$  is the disjoint union of  $X$ -translates of  $\Delta$ .

We also prove that any finite group can be identified in a natural way with a neighbour transitive code.

## 1. INTRODUCTION

The Hamming graph  $\Gamma = H(m, q)$  has vertex set  $V(\Gamma)$  consisting of all  $m$ -tuples over an alphabet  $Q$  of size  $q$  with an edge between two vertices if and only if they differ in exactly one entry. The automorphism group of  $\Gamma$  is the semi-direct product  $\text{Aut}(\Gamma) = B \rtimes L$ , where  $B \cong S_q^m$  and  $L \cong S_m$ . Any block code  $C$  of length  $m$  over  $Q$  can be embedded as a subset of  $V(\Gamma)$ . Two codes in  $\Gamma$  are *equivalent* if there exists an automorphism of  $\text{Aut}(\Gamma)$  that maps one to the other. The distance between two vertices in  $\Gamma$  is the number of entries in which they differ, and the *minimum distance* of  $C$  is the minimum of all distances between distinct codewords of  $C$ .

A *neighbour* of  $C$  is a vertex that is adjacent to some codeword but is not a codeword itself. We denote the set of neighbours of  $C$  by  $C_1$ . A code  $C$  is  *$X$ -neighbour transitive*, or simply *neighbour transitive*, if there exists an  $X \leq \text{Aut}(\Gamma)$  such that both  $C$  and  $C_1$  are  $X$ -orbits. If  $C$  is an  $X$ -neighbour transitive

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code with minimum distance  $\delta \geq 3$ , then there exists a subgroup of  $X$  that has a 2-transitive action on  $Q$  [24, Prop. 2.9], which is therefore either of affine type or almost simple type. If this action is of almost simple type, we say  $C$  is *alphabet almost simple*  $X$ -neighbour transitive. A *frequency permutation array* is a code  $C$  in  $H(m, q)$  where  $m = pq$  for some positive integer  $p$  such that in each codeword of  $C$  every letter in the alphabet  $Q$  appears exactly  $p$  times. In this paper we characterise the following family of neighbour transitive codes.

**Theorem 1.1.** *Let  $C$  be an alphabet almost simple  $X$ -neighbour transitive code with  $K := X \cap B \neq 1$  and minimum distance  $\delta \geq 3$ . Also let  $\text{soc}(K)$  denote the socle of  $K$ , the group generated by the minimal normal subgroups of  $K$ . Then, up to equivalence, there exists a  $\text{soc}(K)$ -orbit  $\Delta$  of  $C$  that is explicitly known, and is described in (7.4), (7.5), or (7.6). Moreover, either  $\Delta = C$ , or  $\Delta$  is a neighbour transitive frequency permutation array and  $C$  is the disjoint union of  $X$ -translates of  $\Delta$ .*

This main result, stated in slightly different terms, can be found in the first author's Ph.D thesis [21, Thm. 6.2]. In Section 2 we introduce the necessary definitions along with some preliminary results. Then we give various constructions of neighbour transitive codes in Section 3. These include the Product and Repetition constructions (Section 3.1) and the Projection codes (Section 3.2). In the next section we describe some concrete examples of neighbour transitive codes, and in particular, we prove Theorem 1.3 (see Section 1.4), which identifies any finite group with a neighbour transitive code. We then turn our attention to proving Theorem 1.1 for an alphabet almost simple  $X$ -neighbour transitive code  $C$ . First in Section 5 we prove that, up to equivalence,  $\text{soc}(K)$  is a sub-direct product of a direct product of non-abelian simple groups (Proposition 5.2). This allows us to apply Scott's Lemma [43], and in turn, determine the structure of  $\text{soc}(K)$ . Using this, in the following section we consider certain Projection codes of  $C$ , and use the classification of diagonally neighbour transitive codes [20] (see Definition 2.1) to describe the codes that can appear. This allows us in Section 7 to describe certain  $\text{soc}(K)$ -orbits in  $C$ . We prove Theorem 1.1 in Section 8. In the final section we give an example of a code that satisfies the conditions of Theorem 1.1, but with the additional property that the projected code has a minimum distance strictly less the minimum distance of the original code. In the remainder of this section we discuss the context of our investigation, in particular how it relates to earlier studies of completely regular codes, and to recent work on codes suitable for powerline communication.

**1.1. The assumption that errors are equally likely.** An ideal decoding decision scheme for communicating across a noisy channel can depend on the probability characteristics of the input [42, p.93]. To combat this, it is often assumed that the input distribution is uniform, that is, each codeword has an equal probability of being sent [30, p.10]. In this circumstance, maximum likelihood decision schemes are suitable [42, p.95]; given an output message  $\mathbf{y}$ , the codeword  $\mathbf{x}$  is chosen that maximises  $p(\mathbf{y}|\mathbf{x})$ , the probability that  $\mathbf{y}$  was received given that  $\mathbf{x}$  was sent. A more descriptive decision scheme is *nearest neighbour decoding* (also known as *minimum distance decoding*); given a received message  $\mathbf{y}$ , a minimum distance decoder will decode to the codeword  $\mathbf{x}$  that is closest to  $\mathbf{y}$  with respect to the distance metric inherent in the Hamming graph. *Syndrom decoding* used for linear codes is essentially nearest neighbour decoding, but the algebraic structure of the code allows for any received vertex to be checked against a shortened list of codewords [28, Sec. 1.11].

If the probability of an error occurring during transmission is independent of the symbol sent, and also, in the case where an error occurs, each of the other  $q - 1$  symbols has an equal chance of appearing, then maximum likelihood decoding is equivalent to nearest neighbour decoding [42, p.130], [27, p.5]. Thus, if we are transmitting across a discrete memoryless channel (the case for which Shannon's Theorem [44] was originally proved), assuming that single errors are equally likely allows one to use nearest neighbour decoding to obtain maximum likelihood decoding. Hence, in coding theory it is often assumed that the probability of each error occurring is independent of both the position in which the error occurs, and the symbol appearing in error [36, p.4], [42, p.122], [37, p.5]. The authors introduced the property of neighbour transitivity as a group theoretic analogue to the assumption that single errors are equally likely [22], and also characterised  $X$ -neighbour transitive codes for certain groups  $X$  [20]. This paper further contributes to the problem of characterising neighbour transitive codes in general.

**1.2. Some historical remarks.** Ever since Shannon's seminal 1948 paper, coding theorists have been interested in codes that are highly structured and symmetrical, the hope being that codes with these properties will have good error correcting capabilities, and at the same time, can be efficiently decoded.

One of the first families of codes that coding theorists studied were *perfect codes*; a code with minimum distance  $\delta$  is *perfect* if the spheres of radius  $e = \lfloor \frac{\delta-1}{2} \rfloor$  centred on the codewords cover the vertices of the Hamming graph. Trivial examples of perfect codes are the code containing just one codeword, the whole space, and the binary *repetition code* where  $m$  is odd (see Example 2.2). Non-trivial perfect codes include the Hamming codes and the perfect Golay codes. Building on work by van Lint, Tietäväinen [48] proved that any non-trivial perfect code over a finite field has the same parameters as either a Hamming code or one of the perfect Golay codes.

Once the parameters of perfect codes had been classified, coding theorists began examining other families of codes with large amounts of structure, including nearly perfect and uniformly packed codes. For uniformly packed codes, the spheres of radius  $e + 1$  around codewords cover the full space, but overlap in a regular way. That is, a code is *uniformly packed* if vertices that are at distance  $e$  from some codeword are in  $\lambda + 1$  spheres, and vertices that are at distance  $e + 1$  or more from every codeword are in  $\mu$  spheres (where  $\lambda$  and  $\mu$  are constants and  $\lambda < (m - e)(q - 1)/(e + 1)$ ). If  $\lambda = \lfloor (m - e)(q - 1)/(e + 1) \rfloor$  and  $\mu = \lfloor m(q - 1)/(e + 1) \rfloor$ , the code is *nearly perfect*.

Lindström [31] classified the parameters of binary nearly perfect codes; the parameters are those of either the punctured Preparata code, or the code constructed by puncturing the codewords of even weight in the binary Hamming code. He also showed that nearly perfect codes over non-binary finite fields are necessarily perfect [32]. In his thesis, van Tilborg [49] proved the non-existence of uniformly packed codes with  $e \geq 4$ , and that the extended binary Golay code is the only binary uniformly packed code with  $e = 3$ . He also classified binary uniformly packed codes with  $\lambda$  and  $\mu$  such that  $\mu - \lambda = 1$  (such codes are called *strongly uniformly packed*). A classification of binary linear uniformly packed codes with  $e = 2$  was given by Calderbank and Goethals [10], with one outstanding case being dealt with by Calderbank [9].

In 1973 Delsarte [15] introduced *completely regular codes*, a family of codes with a high degree of combinatorial symmetry (see Definition 2.3). Delsarte showed that perfect codes are completely regular, and it also holds that uniformly packed codes are completely regular, see for example [47]. Further examples of completely regular codes are the Preparata codes, the Kasami codes, various codes constructed from the Golay codes, and the Hadamard 12 and punctured Hadamard 12 code. However, a belief began to emerge amongst coding theorists that completely regular codes are actually quite rare [33]. Indeed, in his 1990 paper, Neumaier [34] conjectured that the only completely regular codes with minimum distance at least 8 are the binary repetition code and the extended binary Golay code. Surprisingly Borges, Rifà and Zinoviev [6] found a counter example to this conjecture, and have since written a series of papers classifying various families and finding new examples of completely regular codes [7, 39, 40, 41, 50]. In particular, many of their examples are also *completely transitive* (see Definition 2.1), a family of completely regular codes that are very symmetric from an algebraic viewpoint. Currently the classification of completely regular codes is an open problem. A result of Brouwer et al. [8, p.353] shows that certain families of distance-regular graphs are coset graphs of additive completely regular codes, and so a classification of these codes may be of interest to graph theorists as well as coding theorists. With this in mind, a classification of completely transitive codes seems valuable, and as such the authors have classified all  $X$ -completely transitive codes with  $K := X \cap B = 1$  and minimum distance at least 5 [24]. The case  $K \neq 1$  is more difficult, however. As completely transitive codes are necessarily neighbour transitive, any characterisation of neighbour transitive codes is certainly useful in the context of classifying completely transitive codes.

**Remark 1.2.** Many of the classifications mentioned above only hold for  $q$ -ary codes with  $q$  a prime power. In particular, if  $q$  is not a prime power, not even perfect codes have been classified. There are some results though. For example, Best [3] showed that a perfect code over a non-prime power alphabet must have error correcting capability  $e = 1, 2, 6$ , or  $8$ . We note that the classification of completely transitive codes with  $K = 1$  and  $\delta \geq 5$  is over any alphabet size.

**1.3. Connections with powerline communication.** Powerline communication has been proposed as a possible solution to the “last mile problem” in the delivery of telecommunications [26, 35]. By allowing the frequency at which electricity is transmitted over powerlines to vary, say  $q$  distinct frequencies, an alphabet is generated over which information can be encoded [14, 17]. However, it is likely that the power output will not be constant if an arbitrary code is used, interfering with the primary purpose of the electrical infrastructure. There are also additional types of noise that need to be considered for powerline communication. As well as the usual background noise, there is a permanent narrow band noise present generated by electrical equipment and a short term impulse noise that affects many frequencies over a short period of time [13].

Constant composition codes have been suggested as suitable coding schemes to deal with the extra noise considerations present in powerline communication while at the same time providing a constant power output [12, 13, 14]. These are  $q$ -ary codes with the property that the number of occurrences of each symbol within a codeword is the same for each codeword. Frequency permutation arrays are a class of constant composition codes that are ideal, in some sense, for powerline communication [13]. The most

extensively studied frequency permutation arrays (indeed constant composition codes) are permutation codes, see for example [2, 4, 5, 12, 14, 18, 29, 46, 45].

It follows from Theorem 1.1 that frequency permutation arrays arise naturally as the building blocks of certain neighbour transitive codes. In particular, in Section 7 we see that certain neighbour transitive permutation codes are the building blocks in one case, and in a second case, twisted permutation codes (which are also frequency permutation arrays) are the building blocks. Twisted permutation codes were introduced by the authors in [23] and are less well known, but can have improved error correcting capabilities over repeated copies of the usual permutation codes [23, Theorem 1.1].

**1.4. Every finite group can be considered as a neighbour transitive code.** Let  $G$  be a finite group of order  $q$ , and consider the Hamming graph  $\Gamma = H(q, q)$  over  $G$  of length  $q$ , that is  $V(\Gamma)$  consists of all  $q$ -tuples with entries from  $G$ . Cayley's theorem tells us that  $G$  has a faithful regular action on itself by multiplication on the right. Let  $r(g)$  denote the image of  $g \in G$  under this action, and consider a fixed ordering  $\mathfrak{o} = (g_1, \dots, g_q)$  of the elements of  $G$ . Then we can define the following vertex and code in  $H(q, q)$ . For  $g \in G$  define

$$\alpha_{\mathfrak{o}}(g) = (g_1^{r(g)}, \dots, g_q^{r(g)}) = (g_1 g, \dots, g_q g)$$

and the *permutation code of  $G$  with respect to  $\mathfrak{o}$*  to be

$$C_{\mathfrak{o}}(G) = \{\alpha_{\mathfrak{o}}(g) \mid g \in G\}.$$

Given two orderings  $\mathfrak{o}$  and  $\mathfrak{o}'$  of the elements of  $G$ , we prove in Example 4.7 that the codes  $C_{\mathfrak{o}}(G)$  and  $C_{\mathfrak{o}'}(G)$  are equivalent in  $\Gamma$ . Thus we can talk of *the permutation code of  $G$* , which we denote by  $C(G)$ . Let  $S_G$  denote the Symmetric group of  $G$ . We prove the following theorem in Example 4.7.

**Theorem 1.3.** *Let  $G$  be finite group and  $C(G)$  be the permutation code of  $G$ . Then  $C(G)$  is  $(S_G \times S_G)$ -neighbour transitive.*

## 2. DEFINITIONS AND PRELIMINARIES

Recall that any code of length  $m$  over an alphabet  $Q$  of size  $q$  can be embedded in the vertex set of the *Hamming graph*. The Hamming graph  $\Gamma = H(m, q)$  has vertex set  $V(\Gamma)$ , the set of  $m$ -tuples with entries from  $Q$ , and an edge exists between two vertices if and only if they differ in precisely one entry. Throughout we assume that  $m, q \geq 2$ . The automorphism group of  $\Gamma$ , which we denote by  $\text{Aut}(\Gamma)$ , is the semi-direct product  $B \rtimes L$  where  $B \cong S_q^m$  and  $L \cong S_m$ , see [8, Theorem 9.2.1]. Let  $g = (g_1, \dots, g_m) \in B$ ,  $\sigma \in L$  and  $\alpha = (\alpha_1, \dots, \alpha_m) \in V(\Gamma)$ . Then  $g$  and  $\sigma$  act on  $\alpha$  in the following way:

$$(2.1) \quad \alpha^g = (\alpha_1^{g_1}, \dots, \alpha_m^{g_m}), \quad \alpha^\sigma = (\alpha_{1\sigma^{-1}}, \dots, \alpha_{m\sigma^{-1}}).$$

For any subgroup  $T$  of  $S_q$ , we define the following subgroup of  $B$ :

$$(2.2) \quad \text{Diag}_m(T) = \{x_h = (h, \dots, h) \in B : h \in T\}.$$

Let  $M = \{1, \dots, m\}$ , and view  $M$  as the set of vertex entries of  $H(m, q)$ . For all pairs of vertices  $\alpha, \beta \in V(\Gamma)$ , the *Hamming distance* between  $\alpha$  and  $\beta$ , denoted by  $d(\alpha, \beta)$ , is defined to be the number of entries in which the two vertices differ. We let  $\Gamma_k(\alpha)$  denote the set of vertices in  $H(m, q)$  that are at distance  $k$  from  $\alpha$ .

For a code  $C$  in  $H(m, q)$ , the *minimum distance*,  $\delta$ , of  $C$  is the smallest distance between distinct codewords of  $C$ , which we sometimes denote by  $\delta(C)$  if we want to make specific reference to the code. For any  $\gamma \in V(\Gamma)$ , we define

$$d(\gamma, C) = \min\{d(\gamma, \beta) : \beta \in C\}$$

to be the *distance of  $\gamma$  from  $C$* . The *covering radius of  $C$* , which we denote by  $\rho$ , is the maximum distance that any vertex in  $H(m, q)$  is from  $C$ . We let  $C_i$  denote the set of vertices that are at distance  $i$  from  $C$ , and deduce, for  $i \leq \lfloor (\delta - 1)/2 \rfloor$ , that  $C_i$  is the disjoint union of  $\Gamma_i(\alpha)$  as  $\alpha$  varies over  $C$ . Furthermore,  $C_0 = C$  and  $\{C, C_1, \dots, C_\rho\}$  forms a partition of  $V(\Gamma)$  called the *distance partition of  $C$* . In particular, the *complete code*  $C = V(\Gamma)$  has covering radius 0 and trivial distance partition  $\{C\}$ ; and if  $C$  is not the complete code, we call the non-empty subset  $C_1$  the *set of neighbours of  $C$* .

**Definition 2.1.** Let  $C$  be a code with distance partition  $\{C, C_1, \dots, C_\rho\}$ . Recall that  $C$  is  *$X$ -neighbour transitive*, or simply *neighbour transitive*, if there exists  $X \leq \text{Aut}(\Gamma)$  such that  $C$  and  $C_1$  are  $X$ -orbits in  $H(m, q)$ . If  $C$  is  $X$ -neighbour transitive with  $X \leq \text{Diag}_m(S_q) \rtimes L$  then we say  $C$  is *diagonally neighbour transitive*. If each  $C_i$  is an  $X$ -orbit for  $i = 0, \dots, \rho$  we say  $C$  is  *$X$ -completely transitive*, or simply *completely transitive*.

**Example 2.2.** For  $a \in Q$  let  $(a^m) = (a, \dots, a) \in H(m, q)$ . The *repetition code* in  $H(m, q)$  is the code

$$\text{Rep}(m, q) = \{(a^m) : a \in Q\}.$$

It has minimum distance  $\delta = m$  and is one of the simplest neighbour transitive codes [20]. It is also true that  $\text{Rep}(m, 2)$  is completely transitive [24]. However, if  $m \geq 4$  and  $q \geq 3$  then  $\text{Rep}(m, q)$  is not completely transitive [24, Lemma 2.15].

We say two codes  $C$  and  $C'$  are *equivalent* if there exists  $x \in \text{Aut}(\Gamma)$  such that  $C^x = C'$ , and if  $C' = C$  we call  $x$  an automorphism of  $C$ . The *automorphism group of  $C$* , denoted by  $\text{Aut}(C)$ , is the setwise stabiliser of  $C$  in  $\text{Aut}(\Gamma)$ . It turns out that any neighbour transitive code in  $H(m, q)$  with minimum distance  $\delta = m$  is equivalent to  $\text{Rep}(m, q)$ . To explain this result, we introduce  *$s$ -regular* codes.

**Definition 2.3.** A code with covering radius  $\rho$  is  *$s$ -regular*, for a given  $s \in \{0, \dots, \rho\}$ , if for  $k \geq 1$  and  $\nu \in C_i$ , with  $0 \leq i \leq s$ , the cardinality of the set  $\Gamma_k(\nu) \cap C$  is independent of  $\nu$  and only depends on  $i$  and  $k$ . A code that is  $\rho$ -regular is said to be *completely regular*.

It is a consequence of the definitions that neighbour transitive codes are necessarily 1-regular, and it is known that completely transitive codes are completely regular [25]. The next result follows directly from [24, Lemma 2.13].

**Lemma 2.4.** *Let  $C$  be a code in  $H(m, q)$  with  $|C| \geq 2$  and  $\delta = m \geq 2$ . Then there exists  $C'$  equivalent to  $C$  with  $C' \subseteq \text{Rep}(m, q)$ . Moreover if  $C$  is 1-regular then  $C' = \text{Rep}(m, q)$ . In particular, any neighbour transitive code with  $\delta = m$  is equivalent to  $\text{Rep}(m, q)$ .*

Let  $C$  be a code and  $\alpha$  be any vertex in  $H(m, q)$ . As  $\text{Aut}(\Gamma)$  acts transitively on the vertices of  $\Gamma$ , there exists  $y \in \text{Aut}(\Gamma)$  such that  $\alpha \in C^y$ . The next result allows us to take advantage of this fact.

**Lemma 2.5.** *([Lem. 2, [20]]) Let  $C$  be a code with distance partition  $\mathcal{C} = \{C, C_1, \dots, C_\rho\}$  and  $y \in \text{Aut}(\Gamma)$ . Then  $C_i^y := (C_i)^y = (C^y)_i$  for each  $i$ . In particular, the code  $C^y$  has distance partition  $\{C^y, C_1^y, \dots, C_\rho^y\}$ , and  $\mathcal{C}$  is  $\text{Aut}(C)$ -invariant. Moreover,  $C$  is  $X$ -neighbour transitive if and only if  $C^y$  is  $X^y$ -neighbour transitive.*

By replacing  $C$  with the equivalent code  $C^y$  if necessary, Lemma 2.5 allows us to assume that  $\alpha$  is a codeword in our neighbour transitive code  $C$ . We use this trick several times throughout this paper.

**2.1. Description of Neighbours.** Let  $\alpha = (\alpha_1, \dots, \alpha_m)$  be a vertex in  $H(m, q)$ , and for  $a \in Q$  let  $\nu(\alpha, i, a)$  denote the vertex with  $j$ th entry

$$(2.3) \quad \nu(\alpha, i, a)|_j = \begin{cases} \alpha_j & \text{if } j \neq i \\ a & \text{if } j = i. \end{cases}$$

We note that if  $\alpha_i = a$  then  $\nu(\alpha, i, a) = \alpha$ , otherwise it is adjacent to  $\alpha$ . Throughout this paper if we refer to  $\nu(\alpha, i, a)$  as a neighbour of  $\alpha$ , or being adjacent to  $\alpha$ , the reader should assume that  $a \in Q \setminus \{\alpha_i\}$ . For  $x = (h_1, \dots, h_m)\sigma \in \text{Aut}(\Gamma)$  it is known that

$$(2.4) \quad \nu(\alpha, i, a)^x = \nu(\alpha^x, i^\sigma, a^{h_i}),$$

which is neighbour of  $\alpha^x$  if and only if  $\nu(\alpha, i, a)$  is a neighbour of  $\alpha$  [20, Lemma 1]. Combining this with the following result will prove useful in the sequel.

**Lemma 2.6.** *Let  $\nu(\alpha, i, a)$  and  $\nu(\beta, j, b)$  be respective neighbours of  $\alpha = (\alpha_1, \dots, \alpha_m)$  and  $\beta = (\beta_1, \dots, \beta_m)$  in  $H(m, q)$  such that  $\nu(\alpha, i, a) = \nu(\beta, j, b)$ . Then one of the following holds:*

- (i)  $\alpha = \beta$ ,  $i = j$  and  $a = b$ ;
- (ii)  $i = j$ ,  $a = b$  and  $\beta = \nu(\alpha, i, c)$  for some  $c \neq \alpha_i, a$ ;
- (iii)  $d(\alpha, \beta) = 2$ ,  $i \neq j$ ,  $\alpha_j = b$  and  $\beta_i = a$ .

*Proof.* It follows that  $d(\alpha, \beta) \leq d(\alpha, \nu(\alpha, i, a)) + d(\nu(\alpha, i, a), \nu(\beta, j, b)) = 2$  since  $\nu(\alpha, i, a) = \nu(\beta, j, b)$ . Firstly assume  $d(\alpha, \beta) = 0$ . Then  $\alpha = \beta$ . Now suppose  $i \neq j$ . Then  $\nu(\alpha, i, a)|_i = \nu(\beta, j, b)|_i = \alpha_i$  since  $\alpha = \beta$  and  $i \neq j$ . Thus  $\alpha = \nu(\alpha, i, a)$  contradicting the fact that  $\nu(\alpha, i, a)$  is a neighbour of  $\alpha$ . Hence  $i = j$  and so  $\nu(\alpha, i, a)|_i = a = b = \nu(\beta, i, b)|_i$ . Thus (i) holds.

Now assume  $d(\alpha, \beta) = 1$ . Then  $\beta = \nu(\alpha, k, c)$  for some  $c \neq \alpha_k$ . Suppose  $i, j \neq k$ . Then  $\nu(\alpha, i, a)|_k = \alpha_k = \nu(\beta, j, b)|_k = c$ , which is a contradiction. Thus  $i = k$  or  $j = k$ . Suppose  $i = k$  and  $j \neq k$ . Then  $b \neq \beta_j = \alpha_j$ . However,  $\nu(\alpha, k, a)|_j = \alpha_j = \nu(\beta, j, b)_j = b$ , which is a contradiction. The condition  $j = k$  and  $i \neq k$  leads to a similar contradiction. Thus  $i = j = k$ . Hence



$\nu(\alpha, i, a)|_i = a = b = \nu(\beta, i, b)|_i$ . Suppose  $c = a$  and so  $c = b$ . Then  $\nu(\beta, j, b) = \nu(\beta, k, c) = \beta$ , contradicting the fact that  $\nu(\beta, j, b)$  is a neighbour of  $\beta$ . Thus (ii) holds.

Finally, assume  $d(\alpha, \beta) = 2$ . Suppose  $i = j$ . Then  $\nu(\alpha, i, a) = \nu(\beta, i, b)$ , and so by definition,  $\alpha_k = \beta_k$  for all  $k \neq i$ . Thus  $d(\alpha, \beta) \leq 1$  and we have a contradiction. Thus  $i \neq j$ . It follows from the definitions that  $\alpha_j = \nu(\beta, j, b) = b$  and  $\beta_i = \nu(\alpha, i, a) = a$ .  $\square$

**2.2. Group Actions.** For a nonempty set  $\Omega$ , we denote the group of permutations of  $\Omega$  by  $\text{Sym}(\Omega)$ . A *permutation group*  $G$  on  $\Omega$  is a subgroup of  $\text{Sym}(\Omega)$ . The *minimal degree* of  $G$  is the smallest number of points moved by any non-identity element of  $G$ . We say  $G$  acts *regularly* on  $\Omega$  if  $G$  is a transitive subgroup of  $\text{Sym}(\Omega)$  and  $G_\alpha = 1$  for all  $\alpha \in \Omega$ .

For an abstract group  $G$ , an *action* of  $G$  on  $\Omega$  is a homomorphism  $\rho$  from  $G$  to  $\text{Sym}(\Omega)$ . The *degree of the action* is the cardinality of  $\Omega$ . Let  $\rho_1 : G \rightarrow \text{Sym}(\Omega)$  and  $\rho_2 : H \rightarrow \text{Sym}(\Omega')$  be actions of the groups  $G, H$  on  $\Omega, \Omega'$  respectively. We say these actions are *permutationally isomorphic* if there exists a bijection  $\lambda : \Omega \rightarrow \Omega'$  and an isomorphism  $\varphi : \rho_1(G) \rightarrow \rho_2(H)$  such that

$$\lambda(\alpha^{\rho_1(g)}) = \lambda(\alpha)^{\varphi(\rho_1(g))} \quad \text{for all } \alpha \in \Omega \text{ and } g \in G,$$

and we call  $(\lambda, \varphi)$  a *permutational isomorphism*.

We now consider three distinct actions for the automorphism group  $X$  of an  $X$ -neighbour transitive code. First we consider its natural action on the code.

**Lemma 2.7.** *Let  $C$  be an  $X$ -neighbour transitive code with minimum distance  $\delta \geq 3$ . Let  $\Delta$  be a block of imprimitivity for the action of  $X$  on  $C$ . Then  $\Delta$  is an  $X_\Delta$ -neighbour transitive code with minimum distance at least  $\delta$ .*

*Proof.* Since  $\Delta$  is a block of imprimitivity for the action of  $X$  on  $C$ , it follows that  $X_\Delta$  acts transitively on  $\Delta$  [16, p.13]. Let  $\nu_1, \nu_2$  be neighbours of  $\Delta$ . Then there exist codewords  $\alpha_1, \alpha_2$  of  $\Delta$  that are respectively adjacent to  $\nu_1, \nu_2$ . As  $C$  is  $X$ -neighbour transitive, there exists  $x \in X$  such that  $\nu_1^x = \nu_2$ . We claim that  $\alpha_1^x = \alpha_2$ . Suppose not. Then there exists a codeword  $\alpha_3 \in C$  such that  $\alpha_1^x = \alpha_3$  and  $\nu_2$  is a neighbour of  $\alpha_3$ . This implies that  $d(\alpha_2, \alpha_3) \leq 2$ , contradicting the minimum distance of  $C$ . Hence  $\alpha_1^x = \alpha_2$ . Therefore, because  $\Delta$  is a block of imprimitivity,  $\Delta^x = \Delta$ . Finally, because  $C$  has minimum distance  $\delta$  and  $\Delta \subseteq C$ , it follows directly that  $\Delta$  has minimum distance at least  $\delta$ .  $\square$

Let us now describe two alternative actions for an automorphism group  $X \leq \text{Aut}(\Gamma)$ . We define

$$\begin{aligned} \mu : \text{Aut}(\Gamma) &\longrightarrow S_m \\ h\sigma &\longmapsto \sigma, \end{aligned}$$

and when we talk of the action of  $X$  on  $M$ , or the action of  $X$  on entries, we are referring to the action of  $\mu(X)$  on  $M$ . We denote  $\mu(X)$  by  $X^M$ . Now, for  $i \in M$  let  $X_i = \{x = h\sigma \in X : i^\sigma = i\}$ , which has an action on the alphabet  $Q$  via the following homomorphism:

$$(2.5) \quad \begin{aligned} \varphi_i : X_i &\longrightarrow S_q \\ (h_1, \dots, h_m)\sigma &\longmapsto h_i \end{aligned}$$

We denote the image  $\varphi_i(X_i)$  by  $X_i^Q$ . Next we collate three results that appear in [24, Prop. 2.7, Cor. 2.8, Prop. 2.9], applying them to our context.

**Proposition 2.8.** *Let  $C$  be an  $X$ -neighbour transitive code with  $\delta \geq 3$ . Let  $\alpha \in C$  and  $i \in M$ . Then*

- (i)  $X_\alpha$  acts transitively on  $\Gamma_1(\alpha)$  and  $M$ ,
- (ii)  $X_i$  acts transitively on  $C$ , and
- (iii)  $X_i^Q$  acts 2-transitively.

Proposition 2.8 gives us that  $X_1^Q$  is 2-transitive for any  $X$ -neighbour transitive codes with minimum distance  $\delta \geq 3$ . It follows from Burnside's Theorem [16, Theorem 4.1B] that any 2-transitive group is of almost simple type or of affine type. Thus, we recall the following definition from the introduction.

**Definition 2.9.** Let  $C$  be an  $X$ -neighbour transitive code. We say  $C$  is *alphabet almost simple neighbour transitive* if  $X^M$  acts transitively on  $M$  and  $X_1^Q$  is 2-transitive of almost simple type. Similarly, if  $X^M$  acts transitively and  $X_1^Q$  is 2-transitive of affine type, we say  $C$  is *alphabet affine neighbour transitive*.

### 3. CONSTRUCTIONS OF NEIGHBOUR TRANSITIVE CODES

**3.1. Product and Repetition Constructions.** In this section we consider  $\ell$ -tuples of codewords from a code  $C$  in  $H(m, q)$ . Let us first consider the set of all  $\ell$ -tuples of vertices from  $H(m, q)$ , that is,

$$\Gamma^\ell = \{(\alpha_1, \dots, \alpha_\ell) : \alpha_i \in H(m, q)\}.$$

It is clear that we can identify  $\Gamma^\ell$  with the vertex set of  $H(m\ell, q)$ . For any  $X \leq \text{Aut}(\Gamma)$  we define an action of  $X \wr S_\ell$  on  $\Gamma^\ell$  in the natural way. In particular, for  $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \Gamma^\ell$  and  $y = (x_1, \dots, x_\ell)\sigma \in X \wr S_\ell$ ,

$$(3.1) \quad \alpha^y = (\alpha_1^{x_1}, \dots, \alpha_\ell^{x_\ell})^\sigma = (\alpha_{1\sigma^{-1}}^{x_1\sigma^{-1}}, \dots, \alpha_{\ell\sigma^{-1}}^{x_\ell\sigma^{-1}}).$$

Now, for an arbitrary code  $C$  in  $H(m, q)$  we denote the complete set of  $\ell$ -tuples of codewords from  $C$  by

$$\text{Prod}_\ell(C) = \{(\alpha_1, \dots, \alpha_\ell) \mid \alpha_i \in C \text{ for all } i\}.$$

For  $\alpha \in \Gamma^\ell$  and  $\nu \in \Gamma$  we let  $\gamma(\alpha, i, \nu)$  denote the vertex constructed by changing that  $i$ th vertex entry of  $\alpha$  from  $\alpha_i$  to  $\nu$ . It follows that  $\gamma(\alpha, i, \nu)$  is in  $\text{Prod}_\ell(C)_1$ , the neighbour set of  $\text{Prod}_\ell(C)$ , if and only if  $\nu \in C_1$ . Also, all members of  $\text{Prod}_\ell(C)_1$  are of this form.

**Lemma 3.1.** *Let  $C$  be an  $X$ -neighbour transitive code in  $H(m, q)$  with minimum distance  $\delta$ . Then  $\text{Prod}_\ell(C)$  is  $(X \wr S_\ell)$ -neighbour transitive in  $H(m\ell, q)$  with minimum distance  $\delta$ .*

*Proof.* It is clear that  $X \wr S_\ell$  fixes  $\text{Prod}_\ell(C)$  setwise, and so by Lemma 2.5, it also fixes setwise the neighbour set of  $\text{Prod}_\ell(C)$ . Moreover, because  $X$  acts transitively on  $C$  it follows that  $X^\ell$  acts transitively on  $\text{Prod}_\ell(C)$ . Now let  $\gamma(\alpha, i, \nu)$  and  $\gamma(\beta, j, \nu')$  be two neighbours of  $\text{Prod}_\ell(C)$ . As  $S_\ell$  acts transitively on  $\{1, \dots, \ell\}$  and is contained in the automorphism group of  $\text{Prod}_\ell(C)$ , we can assume without loss of generality that  $i = j$ . As both  $C$  and  $C_1$  are  $X$ -orbits in  $H(m, q)$ , there exist  $x_k \in X$  such

that  $\alpha_k^{x_k} = \beta_k$  for each  $k \neq i$ , and  $x_i \in X$  such that  $\nu^{x_i} = \nu'$ . By letting  $y = (x_1, \dots, x_\ell)$ , it follows that  $\gamma(\alpha, i, \nu)^y = \gamma(\beta, i, \nu')$ .  $\square$

An interesting subcode of  $\text{Prod}_\ell(C)$  is the set of all vertices with constant entry. We let

$$\text{rep}_\ell(\alpha) = (\alpha, \dots, \alpha) \in \Gamma^\ell$$

for a vertex  $\alpha \in H(m, q)$ , and we define

$$\text{Rep}_\ell(C) = \{\text{rep}_\ell(\alpha) \mid \alpha \in C\}.$$

It follows from [20, Lemma 5] and Proposition 2.8 that if  $C$  is  $X$ -neighbour transitive with  $\delta \geq 3$ , then  $\text{Rep}_\ell(C)$  is  $(X \times S_\ell)$ -neighbour transitive with minimum distance  $\delta\ell$ .

**3.2. Projection Codes.** Given any code  $C$  we want to describe codes that are somehow “contained” in  $C$  but in a “smaller” Hamming graph. To explain this idea let  $J = \{i_1, \dots, i_k\} \subseteq M$ , with  $i_1 < i_2 < \dots < i_k$ , and define the following map

$$\begin{aligned} \pi_J : \quad H(m, q) &\longrightarrow H(J, q) \\ (\alpha_1, \dots, \alpha_m) &\longmapsto (\alpha_{i_1}, \dots, \alpha_{i_k}) \end{aligned}$$

Note that by  $H(J, q)$  we mean the Hamming graph  $H(|J|, q)$ . For a code  $C$  in  $H(m, q)$  we define the *projected code of  $C$  with respect to  $J$*  to be the set

$$\pi_J(C) = \{\pi_J(\alpha) \in H(J, q) : \alpha \in C\}.$$

We are interested in projected codes of neighbour transitive codes, and therefore we want to obtain some group information when we project. Thus for  $x \in \text{Aut}(\Gamma)_J = \{h\sigma \in \text{Aut}(\Gamma) : J^\sigma = J\}$  we define  $\chi(x)$  to be the map

$$(3.2) \quad \begin{aligned} \chi(x) : \quad H(J, q) &\longrightarrow H(J, q) \\ \pi_J(\alpha) &\longmapsto \pi_J(\alpha^x) \end{aligned}$$

We observe that this map is well defined if and only if  $x \in \text{Aut}(\Gamma)_J$ , and that  $\chi(\text{Aut}(\Gamma)_J) \leq \text{Aut}(H(J, q))$ .

**Remark 3.2.** Let  $x = (h_1, \dots, h_m)\sigma \in \text{Aut}(\Gamma)_J$  where  $J = \{i_1, \dots, i_k\}$ . Let  $\hat{\sigma}$  be the induced action of  $\sigma$  on  $J$ . We claim that  $\chi(x) = (h_{i_1}, \dots, h_{i_k})\hat{\sigma}$ . Let  $\alpha = (\alpha_{i_1}, \dots, \alpha_{i_k}) \in H(J, q)$ . Then there exists  $\beta = (\alpha_1, \dots, \alpha_m) \in H(m, q)$  such that  $\pi_J(\beta) = \alpha$ . Thus  $\alpha^{\chi(x)} = \pi_J(\beta)^{\chi(x)} = \pi_J(\beta^x)$ , and so

$$\chi_J(\beta^x) = (\alpha_{i_1\sigma^{-1}}^{h_{i_1\sigma^{-1}}}, \dots, \alpha_{i_k\sigma^{-1}}^{h_{i_k\sigma^{-1}}}).$$

By definition,  $i^\sigma = i^{\hat{\sigma}}$  for all  $i \in J$ . Therefore, since  $J^\sigma = J$ , it follows that  $\chi_J(\beta^x) = \alpha^{(h_{i_1}, \dots, h_{i_k})\hat{\sigma}}$ . Because  $\alpha$  was arbitrarily chosen, we conclude that  $\chi(x) = (h_{i_1}, \dots, h_{i_k})\hat{\sigma}$ .

Let  $\pi_J(C)$  be the projected code in  $H(J, q)$  of  $C$  with distance partition  $\{\pi_J(C), \pi_J(C)_1, \dots, \pi_J(C)_{\rho_J}\}$ . In order to examine how the distance partition of  $\pi_J(C)$  relates to the distance partition of  $C$  we introduce the following set:

$$C_1(J) = \{\nu(\alpha, j, b) \in C_1 : \alpha \in C, j \in J \text{ and } b \in Q \setminus \{\alpha_j\}\}.$$

**Lemma 3.3.** *Let  $C$  be a code in  $H(m, q)$  and  $J \subseteq M$ . Then*

- (i) for all  $\nu \in C_i$ ,  $\pi_J(\nu) \in \pi_J(C)_k$  for some  $k \leq i$ .
- (ii) If  $\delta \geq 2$ , then  $\pi_J(C)_1 \subseteq \pi_J(C_1(J))$ .

*Proof.* (i) For  $\nu \in C_i$ , there exists  $\alpha \in C$  such that  $d(\alpha, \nu) = i$ , and so  $d(\pi_J(\alpha), \pi_J(\nu)) \leq i$ . Hence  $d(\pi_J(\nu), \pi_J(C)) \leq i$ .

(ii) Let  $\nu = (\nu_{i_1}, \dots, \nu_{i_k}) \in \pi_J(C)_1$ . Then  $\nu$  is the neighbour of a codeword  $\pi_J(\alpha)$  for some  $\alpha = (\alpha_1, \dots, \alpha_m) \in C$ . Thus there exists  $j \in J$  such that  $\nu_i = \pi_J(\alpha)_i = \alpha_i$  for all  $i \in J \setminus \{j\}$  and  $\nu_j \neq \pi_J(\alpha)_j = \alpha_j$ . Consider the vertex  $\nu(\alpha, j, \nu_j) \in H(m, q)$ . Since  $\nu_j \neq \alpha_j$  it follows that  $\nu(\alpha, j, \nu_j)$  is adjacent to  $\alpha$ . Moreover,  $\nu(\alpha, j, \nu_j) \in C_1$  since  $\delta \geq 2$ . Because  $j \in J$ , it follows that  $\nu(\alpha, j, \nu_j) \in C_1(J)$ . In addition we have that  $\pi_J(\nu(\alpha, j, \nu_j)) = \nu$ . Thus  $\nu \in \pi_J(C_1(J))$ .  $\square$

We observe that the reverse inclusion of Lemma 3.3–(ii) does not always hold. For example, let  $C$  be a code with  $\delta = 2$  and  $\alpha, \beta \in C$  such that  $d(\alpha, \beta) = 2$  with  $\alpha, \beta$  differing in entries  $i, k \in M$ . Let  $J$  be a proper subset of  $M$  that contains  $i$  but not  $k$ . Consider the vertex  $\nu = \nu(\alpha, i, \beta_i)$ , which is adjacent to  $\alpha$ . Since  $\delta = 2$  and  $i \in J$ , it follows that  $\nu \in C_1(J)$ . However,  $\pi_J(\nu) = \pi_J(\beta) \in \pi_J(C)$ . We now show that, given certain conditions, including  $\delta \geq 3$ , the projected code of a neighbour transitive code is also neighbour transitive.

**Proposition 3.4.** *Let  $C$  be an  $X$ -neighbour transitive code with  $\delta \geq 3$ . Moreover, let  $\mathcal{J} = \{J_1, \dots, J_\ell\}$  be an  $X$ -invariant partition of  $M$ . Then  $\pi_J(C)$  is a  $\chi(X_J)$ -neighbour transitive code for each  $J \in \mathcal{J}$ .*

*Proof.* Let  $\alpha \in C$  and  $x \in X_J$ . As  $\alpha^x \in C$  it follows that  $\pi_J(\alpha)^{\chi(x)} = \pi_J(\alpha^x) \in \pi_J(C)$ . Hence  $\chi(X_J)$  is an automorphism group of  $\pi_J(C)$ . Now, because  $\delta \geq 3$ , it follows from Proposition 2.8–(ii) that, for  $j \in J$ ,  $X_j$  acts transitively on  $C$ . Thus, as  $\mathcal{J}$  is an  $X$ -invariant partition of  $M$ ,  $X_j \leq X_J$ , and so  $X_J$  acts transitively on  $C$ . From this we deduce that  $\chi(X_J)$  acts transitively on  $\pi_J(C)$ .

As  $\chi(X_J)$  is an automorphism group of  $\pi_J(C)$ , Lemma 2.5 implies that  $\chi(X_J)$  fixes setwise  $\pi_J(C)_1$ , the set of neighbours of  $\pi_J(C)$ . Now let  $\nu_1, \nu_2 \in \pi_J(C)_1$ . By Lemma 3.3, there exists  $\alpha, \beta \in C$ ,  $i, j \in J$ ,  $b, c \in Q$  such that  $\nu(\alpha, i, b), \nu(\beta, j, c) \in C_1(J)$  and  $\pi_J(\nu(\alpha, i, b)) = \nu_1$ ,  $\pi_J(\nu(\beta, j, c)) = \nu_2$ . Since  $C$  is  $X$ -neighbour transitive, there exists  $x = h\sigma \in X$  such that  $\nu(\alpha, i, b)^{h\sigma} = \nu(\beta, j, c)$ . As  $\delta \geq 3$ , it follows from (2.4) and Lemma 2.6 that  $i^\sigma = j$ . Therefore, because  $\mathcal{J}$  is an  $X$ -invariant partition of  $M$ , it follows that  $x \in X_J$ . Thus

$$\nu_1^{\chi(x)} = \pi_J(\nu(\alpha, i, b)^x) = \pi_J(\nu(\beta, j, c)) = \nu_2.$$

Hence  $\pi_J(C)$  is  $\chi(X_J)$ -neighbour transitive.  $\square$

We saw in Lemma 3.3–(ii) that if  $\delta \geq 2$  then  $\pi_J(C)_1 \subseteq \pi_J(C_1(J))$ . We now investigate the reverse inclusion, supposing that the conditions of Proposition 3.4 hold.

**Lemma 3.5.** *Let  $C$  be an  $X$ -neighbour transitive code with  $\delta \geq 3$ . Moreover let  $\mathcal{J} = \{J_1, \dots, J_\ell\}$  be an  $X$ -invariant partition of  $M$ . Then, for each  $J \in \mathcal{J}$ , either  $\pi_J(C)$  is the complete code or  $\pi_J(C)_1 = \pi_J(C_1(J))$ .*

*Proof.* Let  $\nu(\alpha, j, b) \in C_1(J)$  and  $x = h\sigma \in X_J$ . Then by (2.4),  $\nu(\alpha, j, b)^x = \nu(\alpha^x, j^\sigma, b^{h_j})$ . As both  $C$  and  $C_1$  are  $X$ -orbits,  $\nu(\alpha^x, j^\sigma, b^{h_j}) \in C_1$  with  $\alpha^x \in C$ , and because  $x \in X_J$ ,  $j^\sigma \in J$ . In particular,  $\nu(\alpha^x, j^\sigma, b^{h_j}) \in C_1(J)$ , that is,  $X_J$  stabilises  $C_1(J)$ .

Now let  $\nu(\beta, i, c) \in C_1(J)$ . It follows from the neighbour transitivity of  $C$  that there exists  $x = h\sigma \in X$  such that  $\nu(\alpha, j, b)^x = \nu(\beta, i, c)$ . As  $\delta \geq 3$ , Lemma 2.6 implies that  $j^\sigma = i$ , and so  $x \in X_J$ . Hence  $X_J$  acts transitively on  $C_1(J)$ , from which it naturally follows that  $\pi_J(C_1(J))$  is a  $\chi(X_J)$ -orbit. By Lemma 3.3 and Proposition 3.4,  $\pi_J(C)_1$  is a subset of  $\pi_J(C_1(J))$  that is also a  $\chi(X_J)$ -orbit, and so we deduce that either  $\pi_J(C)_1 = \pi_J(C_1(J))$  or  $\pi_J(C)_1 = \emptyset$ . The latter case holds if and only if  $\pi_J(C)$  is the complete code in  $H(J, q)$ .  $\square$

The next two results give us a lower bound on the minimum distance of a projected code of a neighbour transitive code for which the conditions of Proposition 3.4 hold.

**Lemma 3.6.** *Let  $C$  be an  $X$ -neighbour transitive code with  $\delta \geq 3$  and  $\mathcal{J} = \{J_1, \dots, J_\ell\}$  be an  $X$ -invariant partition of  $M$ . Let  $J, J' \in \mathcal{J}$ . Then the action of  $\chi(X_J)$  on  $\pi_J(C)$  is permutationally isomorphic to the action of  $\chi(X_{J'})$  on  $\pi_{J'}(C)$ . Moreover,  $\delta(\pi_J(C)) = \delta(\pi_{J'}(C))$ .*

*Proof.* As  $C$  is  $X$ -neighbour transitive with  $\delta \geq 3$ , it follows from Proposition 2.8–(ii) that there exists  $y = h\sigma \in X$  such that  $J^\sigma = J'$ . Define the map  $\lambda_y : \pi_J(C) \rightarrow \pi_{J'}(C)$  given by  $\pi_J(\alpha) \mapsto \pi_{J'}(\alpha^y)$ . As  $y \in \text{Aut}(C)$ , it is clear that  $\lambda_y$  maps onto  $\pi_{J'}(C)$ . Moreover, for  $\alpha, \beta \in C$ ,

$$\begin{aligned} \pi_{J'}(\alpha^y) = \pi_{J'}(\beta^y) &\iff \alpha^y|_k = \beta^y|_k && \forall k \in J' \\ &\iff \alpha_{k\sigma^{-1}}^{h_{k\sigma^{-1}}} = \beta_{k\sigma^{-1}}^{h_{k\sigma^{-1}}} && \forall k \in J' \\ &\iff \alpha_i^{h_i} = \beta_i^{h_i} && \forall i \in J \quad (\text{as } J^\sigma = J') \\ &\iff \alpha_i = \beta_i && \forall i \in J \\ &\iff \pi_J(\alpha) = \pi_J(\beta), \end{aligned}$$

so  $\lambda_y$  is a bijection from  $\pi_J(C)$  to  $\pi_{J'}(C)$ . The map  $\varphi_y : \chi(X_J) \rightarrow \chi(X_{J'})$  given by  $\chi(x) \mapsto \chi(y^{-1}xy)$  is an isomorphism, and one can deduce that  $(\lambda_y, \varphi_y)$  is a permutational isomorphism from the action of  $\chi(X_J)$  on  $\pi_J(C)$  to the action of  $\chi(X_{J'})$  on  $\pi_{J'}(C)$ .

The argument above shows that  $\alpha^y|_k = \beta^y|_k$  for  $k \in J'$  if and only if  $\alpha_{k\sigma^{-1}} = \beta_{k\sigma^{-1}}$  for  $k\sigma^{-1} \in J$ . Hence  $d(\pi_J(\alpha), \pi_J(\beta)) = d(\pi_{J'}(\alpha^y), \pi_{J'}(\beta^y))$ . Now let  $\delta_J = \delta(\pi_J(C))$  and  $\delta_{J'} = \delta(\pi_{J'}(C))$ . By definition, there exist  $\alpha, \beta \in C$  such that  $d(\pi_J(\alpha), \pi_J(\beta)) = \delta_J$ . Therefore  $d(\pi_{J'}(\alpha^y), \pi_{J'}(\beta^y)) = \delta_J$  and so  $\delta_J \geq \delta_{J'}$ . Similarly it follows that  $\delta_{J'} \geq \delta_J$ . Hence  $\delta_J = \delta_{J'}$ .  $\square$

**Corollary 3.7.** *Let  $C$  be an  $X$ -neighbour transitive code with  $\delta \geq 3$  and  $\mathcal{J}$  be an  $X$ -invariant partition of  $M$  with  $J \in \mathcal{J}$ . If  $\pi_J(C)$  is not the complete code then  $\delta(\pi_J(C)) \geq 2$ .*

*Proof.* Suppose  $\delta(\pi_J(C)) = 1$ . Then there exist  $\alpha, \beta \in C$  such that  $d(\pi_J(\alpha), \pi_J(\beta)) = 1$ . In particular, there exists  $k \in J$  such that  $\alpha_k \neq \beta_k$  and  $\alpha_j = \beta_j$  for all  $j \in J \setminus \{k\}$ . Let  $\nu = \nu(\alpha, k, \beta_k)$ , which because  $\delta \geq 3$  and  $k \in J$  is an element of  $C_1(J)$ . If  $\pi_J(C)$  is not the complete code then Lemma 3.5 implies that  $\pi_J(\nu) \in \pi_J(C)_1$ . However  $\pi_J(\nu) = \pi_J(\beta) \in \pi_J(C)$  which is a contradiction, so  $\delta(\pi_J(C)) \geq 2$ .  $\square$

## 4. EXAMPLES OF NEIGHBOUR TRANSITIVE CODES

We now give some further examples of neighbour transitive codes.

**Example 4.1.** Let  $m < q$  and

$$C = \{(\alpha_1, \dots, \alpha_m) \in H(m, q) : \alpha_i \neq \alpha_j \text{ for } i \neq j\}.$$

Then  $C$  has minimum distance  $\delta = 1$  and covering radius  $\rho = m - 1$ .

**Example 4.2.** If 0 is a distinguished letter of  $Q$ , the *weight* of a vertex  $\alpha$  is the number of entries not equal to 0. Let  $m \geq 3$  be odd and  $Q = \{0, 1\}$ . Define

$$C = \{\alpha \in H(m, 2) : \alpha \text{ has weight } \frac{m-1}{2} \text{ or weight } \frac{m+1}{2}\}.$$

Then  $C$  has minimum distance  $\delta = 1$  and covering radius  $\rho = (m - 1)/2$ .

**Example 4.3.** Let  $m = pq$  for some positive integer  $p$ ,  $Q = \{a_1, \dots, a_q\}$ , and  $\alpha = (a_1^p, a_2^p, \dots, a_q^p)$ . We define

$$\text{All}(pq, q) = \alpha^L,$$

the orbit of  $\alpha$  under the top group of  $\text{Aut}(\Gamma)$ . The code  $\text{All}(pq, q)$  has minimum distance  $\delta = 2$  and covering radius  $\rho = p(q - 1)$ .

In [20] the authors proved that, along with the repetition code  $\text{Rep}(m, q)$ , each of the above examples is neighbour transitive with  $\text{Aut}(C) = \text{Diag}_m(S_q) \rtimes L$ , hence also diagonally neighbour transitive. With  $X = \text{Aut}(C)$  it follows that  $X^M = S_m$ ,  $K = X \cap B = \text{Diag}_m(S_q)$  and  $X_1^Q = S_q$  in each case. Thus the code in Example 4.2 is alphabet affine neighbour transitive, where as if  $q \geq 5$ , the codes in the remaining examples are alphabet almost simple neighbour transitive. However,  $\text{Rep}(m, q)$  with  $m \geq 3$  is the only code among these examples that has minimum distance  $\delta \geq 3$ . We observe in this case that  $\text{soc}(K) = \text{Diag}_m(A_q)$  is transitive on  $\text{Rep}(m, q)$ . (Recall that  $\text{soc}(G)$ , the socle of a group  $G$ , is the group generated by the minimal normal subgroups of  $G$ .)

In the same paper the authors proved that any diagonally neighbour transitive code  $C$  in  $H(m, q)$  is either one of the codes in Examples 2.2, 4.1, 4.2, or  $m = pq$  for some positive integer  $p$  and  $C$  is contained in  $\text{All}(pq, q)$ , that is,  $C$  is a frequency permutation array. Recall from Section 1.3 that frequency permutation arrays have been studied recently due to possible applications to powerline communication, with particular interest in permutation codes, the case where  $p = 1$ . Such codes give rise to further examples of neighbour transitive codes.

**Example 4.4.** To describe permutation codes we identify the alphabet  $Q$  with the set  $\{1, \dots, q\}$  and consider codes in the Hamming graph  $\Gamma = H(q, q)$ . For  $g \in S_q$  we define the vertex

$$\alpha(g) = (1^g, \dots, q^g) \in V(\Gamma).$$

For  $y \in S_q$  we let  $x_y = (y, \dots, y) \in B \cong S_q^q$ , and we let  $\sigma(y)$  denote the automorphism induced by  $y$  in  $L \cong S_q$ . It is known (see [20, Lem. 8]) that for all  $g, y \in S_q$ ,

$$\alpha(g)^{x_y} = \alpha(gy) \text{ and } \alpha(g)^{\sigma(y)} = \alpha(y^{-1}g).$$

Now, for a subset  $T \subseteq S_q$ , we define the *permutation code generated by  $T$*  to be the code

$$(4.1) \quad C(T) = \{\alpha(g) \in V(\Gamma) : g \in T\}.$$

In [20, Lem. 9 and Rem. 2] the authors showed that  $C(T)$  is a neighbour transitive code with  $\delta = 2$  if and only if  $T = S_q$ . Thus, if  $T \neq S_q$ , it holds that any neighbour transitive  $C(T)$  has minimum distance  $\delta \geq 3$ . In the same paper the authors also proved that if  $T$  is a subgroup of  $S_q$  then  $C(T)$  is diagonally neighbour transitive if and only if  $N_{S_q}(T)$  is 2-transitive [20, Thm. 2]. In particular, let

$$A(T) = \{a_t = x_t \sigma(t) : t \in N_{S_q}(T)\}$$

and  $X = \langle \text{Diag}_q(T), A(T) \rangle$ . It is shown in [20, proof of Thm. 2] that, if  $N_{S_q}(T)$  is 2-transitive, then  $C(T)$  is  $X$ -neighbour transitive.

Suppose now that  $N_{S_q}(T)$  is 2-transitive. Then the intersection  $K$  of the group  $X$  with the base group  $B$  is  $K = \text{Diag}_q(T)$ , and so  $\text{soc}(K) = \text{Diag}_q(\text{soc}(T))$ . It is clear that  $K$  is a normal subgroup of  $X_1$ , so  $T \cong K^Q \trianglelefteq X_1^Q$ , and  $X_1^Q \leq N_{S_q}(T)$ . Thus, if  $T \neq S_q$  and  $N_{S_q}(T)$  is 2-transitive of almost simple type, then  $C(T)$  and  $X$  satisfy the conditions of Theorem 1.1.

In this case, if  $T$  is simple then  $\text{soc}(K)$  acts transitively on  $C(T)$ . If  $T$  is not simple then  $\Delta = \alpha(1)^{\text{soc}(K)} = C(\text{soc}(T))$  is properly contained in  $C(T)$ . Let  $\mathcal{T}$  be a transversal for  $\text{soc}(T)$  in  $T$ . Then it follows that

$$C(T) = \bigcup_{t \in \mathcal{T}} \Delta^{x_t} = \bigcup_{t \in \mathcal{T}} C(\text{soc}(T)t).$$

Note that if  $T = S_q$ ,  $C(T)$  is alphabet almost simple neighbour transitive and  $C(T)$  is the disjoint union of two  $\text{soc}(K)$  orbits, but because  $\delta = 2$  the conditions of Theorem 1.1 are not met.

**Example 4.5.** As in Example 4.4, identify  $Q$  with the set  $\{1, \dots, q\}$ . The following lemma proves that for any regular subgroup  $T$  of  $S_q$ , the code  $C(T)$  from (4.1) is equivalent to  $\text{Rep}(q, q)$ , and hence neighbour transitive by Lemma 2.5.

**Lemma 4.6.** *Let  $T$  be a non-trivial subgroup of  $S_q$ , and let  $C(T)$  be as (4.1) with minimum distance  $\delta$ . Then (i)  $\delta = q$  if and only if  $T$  acts semi-regularly on  $Q$ ; and (ii)  $C(T)$  is equivalent to  $\text{Rep}(q, q)$  if and only if  $T$  acts regularly on  $Q$ .*

*Proof.* (i) Assume that  $\delta = q$ . Suppose there exists  $i \in Q$  such that the stabiliser  $T_i \neq 1$ . Then there exists  $1 \neq g \in T_i$  such that  $\alpha(1)|_i = \alpha(g)|_i = i$ . In particular,  $d(\alpha(1), \alpha(g)) \leq q - 1$ , contradicting the fact that  $\delta = q$ . Therefore  $T$  acts semi-regularly on  $Q$ . Now assume that  $T_i = 1$  for all  $i \in Q$ . Consider the distinct vertices  $\alpha(g_1), \alpha(g_2) \in C(T)$  such that  $d(\alpha(g_1), \alpha(g_2)) = \delta$ . If  $\delta < q$  then  $\alpha(g_1)|_i = \alpha(g_2)|_i$  for some  $i \in Q$ , and so  $i^{g_1} = i^{g_2}$ . Thus  $i^{g_1 g_2^{-1}} = i$ , contradicting the fact that  $T_i = 1$  for all  $i \in Q$ .

(ii) Suppose that  $C(T)$  is equivalent  $\text{Rep}(q, q)$ . Then  $C(T)$  has minimum distance  $\delta = q$ , so by (i),  $T_i = 1$  for all  $i \in Q$ . Furthermore  $|T| = |C(T)| = |\text{Rep}(q, q)| = |Q| = q$ . Thus, the orbit stabiliser theorem implies that  $T$  acts transitively on  $Q$ . Hence  $T$  acts regularly on  $Q$ . Conversely suppose that  $T$  acts regularly on  $Q$ . Then by (i),  $C(T)$  has minimum distance  $\delta = q$ . Hence by Lemma 2.4,  $C(T)$  is equivalent to a code  $C' \subseteq \text{Rep}(q, q)$ . However, as  $T$  acts regularly on  $Q$ , it follows that  $q = |T| = |C(T)| = |C'|$ , so  $C' = \text{Rep}(q, q)$ .  $\square$

Line	Degree = $q$	$T$	Conditions	$\delta(C(T, T^\tau))$
1	6	$A_6 \trianglelefteq T \leq S_6$	–	8
2	11	$\text{PSL}(2, 11)$	–	16
3	12	$M_{12}$	–	16
4	15	$A_7$	–	24
5	176	$HS$	–	320
6	$(r^n - 1)/(r - 1)$	$\text{PSL}(n, r) \trianglelefteq T \leq \text{PTL}(n, r)$	$n > 2$	$> \sqrt{q} - 1$

TABLE 1. 2-transitive almost simple groups with two inequivalent actions.

**Example 4.7.** Let  $G$  be a finite group of order  $q$  and  $\mathfrak{o} = (g_1, \dots, g_q)$  be a fixed ordering of the elements of  $G$ . Recall from Section 1.4 the code  $C_{\mathfrak{o}}(G)$  in  $\Gamma = H(q, q)$ , the permutation code of  $G$  with respect to  $\mathfrak{o}$ . In the definition of  $C_{\mathfrak{o}}(G)$ , we identify the identity element of  $G$  with the vertex in  $H(q, q)$  associated with the ordering  $\mathfrak{o}$ , namely we define  $\alpha_{\mathfrak{o}}(1) = (g_1, \dots, g_q) \in H(q, q)$ . For another ordering  $\mathfrak{o}' = (g'_1, \dots, g'_q)$  of the elements of  $G$ , there exists  $\sigma \in L \cong S_G$  (the top group of  $\text{Aut}(H(q, q))$ ) such that  $\alpha_{\mathfrak{o}}(1)^\sigma = (g_{1\sigma^{-1}}, \dots, g_{q\sigma^{-1}}) = (g'_1, \dots, g'_q) = \alpha_{\mathfrak{o}'}(1)$  (see (2.1) for the action of  $L$  on vertices). For any  $g \in G$ , we have

$$\alpha_{\mathfrak{o}}(g)^\sigma = (g_1g, \dots, g_qg)^\sigma = (g'_1g, \dots, g'_qg) = \alpha_{\mathfrak{o}'}(g),$$

and so  $C_{\mathfrak{o}}(G)^\sigma = C_{\mathfrak{o}'}(G)$ . Thus, permutation codes of  $G$  with respect to different orderings of  $G$  are equivalent, and we may therefore talk of  $C(G)$  as *the* permutation code of  $G$ . We can now prove Theorem 1.3, which states that  $C(G)$  is  $(S_G \times S_G)$ -neighbour transitive.

*Proof of Theorem 1.3.* Up to equivalence, the permutation code of  $G$  is the code  $C(r(G))$ , where  $r(G) \leq S_G$  is the right regular representation of  $G$ . As  $r(G)$  acts regularly on  $G$ , it follows from Lemma 4.6–(ii) that  $C(G)$  is equivalent to  $\text{Rep}(q, q)$ . In the discussion following Example 4.3, we saw that  $\text{Rep}(q, q)$  is  $(\text{Diag}_q(S_G) \rtimes L)$ -neighbour transitive. Because  $\text{Diag}_q(S_G) \rtimes L \cong S_G \times S_G$ , the fact that  $C(G)$  is  $(S_G \times S_G)$ -neighbour transitive follows from Lemma 2.5.

**Example 4.8.** Let  $T$  be an almost simple 2-transitive permutation group of degree  $q$ , with socle  $S$  (so  $T \leq N_{S_q}(S)$ ), such that  $N_{S_q}(S)$  is a proper subgroup of  $\text{Aut}(S)$ . By the classification of finite 2-transitive groups,  $T$ ,  $q$  are as in one of the lines of Table 1 and  $N_{S_q}(S)$  is a subgroup of index 2 in  $\text{Aut}(S)$  (see, for example, [11, Table 7.4]). Moreover, there exists an outer automorphism  $\tau$  of  $T$  such that  $\tau^2 = 1$ . For  $t \in T$ , we consider the ordered pair

$$\alpha(t, t^\tau) = (\alpha(t), \alpha(t^\tau)) = (1^t, \dots, q^t, 1^{t^\tau}, \dots, q^{t^\tau}),$$

and the code

$$C(T, T^\tau) = \{\alpha(t, t^\tau) : t \in T\},$$

which is a code in  $\Gamma^2$  where  $\Gamma = H(q, q)$ . Let  $(\alpha, \beta)$  be a general element of  $\Gamma^2$  and let  $\sigma$  be the automorphism in the top group that maps  $(\alpha, \beta)$  to  $(\beta, \alpha)$ . Also let

$$\text{Diag}_m(T, T^\tau) = \{(x_t, x_{t^\tau}) \in \text{Diag}_m(T)^2\}$$



and

$$A(T, T^\tau) = \{(a_t, a_{t^\tau}) \in A(T)^2\},$$

acting on  $\Gamma^2$  as in (3.1). By letting  $X = \langle \text{Diag}_m(T, T^\tau), A(T, T^\tau), \sigma \rangle$ , it follows that  $C(T, T^\tau)$  is  $X$ -neighbour transitive [23, Thm. 4.2].

In this case,  $K = \text{Diag}_q(T, T^\tau)$  and  $\text{soc}(K) = \text{Diag}_q(\text{soc}(T), \text{soc}(T)^\tau)$ . As in Example 4.4, we have that  $T \cong K^Q \trianglelefteq X_1^Q \leq N_{S_q}(T)$ . Moreover,  $N_{S_q}(T)$  is 2-transitive of almost simple type for each of the groups in Table 1. Therefore, if  $C(T, T^\tau)$  has minimum distance at least 3, then the code along with the group  $X$  satisfy the conditions of Theorem 1.1. Let us consider the minimum distance of  $C(T, T^\tau)$ .

For  $T$  as in line 1 or 3 of Table 1, the minimum distance of  $C(T, T^\tau)$  is given in [23]. For  $T$  as in lines 2, 4–6 of Table 1,

$$\delta(C(T, T^\tau)) = 2 \times \delta(C(T)),$$

and the minimum distance of  $C(T)$  is equal to the minimal degree of  $T$  [23, Sec. 4.4 & 4.5]. Consequently, the lower bound for the minimum distance given in line 6 of Table 1 follows from the fact that the minimal degree of a primitive permutation group of degree  $q$  that does not contain  $A_q$  is greater than  $\frac{1}{2}(\sqrt{q} - 1)$  [1, Thm. 6.14]. Therefore, it follows from Table 1 that  $C(T, T^\tau)$  has minimum distance at least 3, the only possible exception being the group  $T$  in line 6 with  $(n, r) = (3, 2)$ . However, it is straightforward to verify, using GAP [19], that the minimal degree of  $\text{PSL}(3, 2)$  is 4, and so  $\delta(C(T, T^\tau)) = 8$  in this case. This confirms that  $C(T, T^\tau)$  and  $X$  satisfy the conditions of Theorem 1.1.

For the groups  $T$  in Table 1 that are simple, it holds that  $\text{soc}(K)$  acts transitively on  $C(T, T^\tau)$ . If  $T = S_6$  or  $T$  is as in line 6 of Table 1 with  $\text{PSL}(n, r)$  a proper subgroup of  $T$ , then  $\Delta = \alpha(1, 1)^{\text{soc}(K)}$  is properly contained in  $C(T, T^\tau)$ . It then follows that

$$C(T, T^\tau) = \bigcup_{t \in \mathcal{T}} \Delta^{(x_t, x_{t^\tau})} = \bigcup_{t \in \mathcal{T}} C(\text{soc}(T)t, (\text{soc}(T)t)^\tau)$$

for a transversal  $\mathcal{T}$  of  $\text{soc}(T)$  in  $T$ .

## 5. THE STRUCTURE OF $\text{soc}(K)$

Let  $C$  be an  $X$ -neighbour transitive code with  $\delta \geq 3$  and  $K := X \cap B \neq 1$ , and let

$$\hat{Y} := \varphi_1(X_1) = X_1^Q,$$

where  $\varphi_1$  is as in (2.5). Recall from Proposition 2.8 that  $\hat{Y}$  acts 2-transitively on the alphabet  $Q$ , and is of almost simple type or of affine type [16, Thm. 4.1B]. In particular,  $\hat{Y}$  has a unique minimal normal subgroup  $T$  which is either non-abelian simple or elementary abelian. For the rest of this paper we assume that  $T$  is non-abelian simple, so  $\hat{Y}$  is of almost simple type. In this section we determine the structure of the socle of  $K$ , which we denote by  $\text{soc}(K)$ . In particular, we prove, up to equivalence, that  $\text{soc}(K)$  is a sub-direct product of  $T^m$ , which then enables us to use Scott's Lemma [43] to determine the structure of  $\text{soc}(K)$  more explicitly.

First we introduce the following theorem, which was proved by the second author with Schneider [38, Theorem 1.1–(a)]. The reader should note that we have modified the notation of the original statement to suit our purposes.

**Theorem 5.1.** *Let  $X \leq \text{Aut}(\Gamma) = B \rtimes L$  and  $\alpha \in H(m, q)$ . Suppose that  $X^M$  is transitive on  $M$ . Then there exists  $y \in B$  such that  $X^y \leq X_1^Q \text{ wr } X^M$ . Moreover, if  $X_1^Q$  is transitive on  $Q$ ,  $y$  may be chosen so that  $\alpha^y = \alpha$ .*

It is a consequence of Proposition 2.8 and Theorem 5.1 that, for our  $X$ -neighbour transitive code  $C$ , there exists  $y \in \text{Aut}(\Gamma)$  such that  $X^y \leq \hat{Y} \text{ wr } L$ . Therefore, by replacing  $C$  with an equivalent code if necessary, Lemma 2.5 allows us to assume that  $C$  is  $X$ -neighbour transitive with  $X \leq \hat{Y} \text{ wr } L$  and  $K := X \cap \hat{Y}^m \neq 1$ .

Since  $K$  is contained in the base group of  $\text{Aut}(\Gamma)$  it follows that  $K \leq X_i$  for each  $i$ . We also note that

$$K \leq \prod_{i=1}^m K_i^Q$$

where  $K_i^Q := \varphi_i(K) \leq \hat{Y}$ . Now, because  $K$  is a normal subgroup of  $X$ , and because  $X$  acts transitively on  $M$ , we can prove that each  $K_i^Q$  is conjugate to  $K_1^Q$  in  $\hat{Y}$ . Thus there exists  $x = (g_1, \dots, g_m) \in \hat{Y}^m$  where each  $g_i$  conjugates  $K_i^Q$  to  $K_1^Q$ . By replacing  $C$  by  $C^x$  if necessary, Lemma 2.5 allows us to assume without loss of generality that

$$K \leq Y^m,$$

where  $Y = K_1^Q$ . If  $Y = 1$  then  $K = 1$ , contradicting our assumption. Therefore, because  $K$  is a normal subgroup of  $X_1$ , it follows that  $Y$  is a non-trivial normal subgroup of  $\hat{Y}$ , and so, as  $\hat{Y}$  is almost simple,  $T$  is also the unique minimal normal subgroup of  $Y$ .

**Proposition 5.2.** *Let  $G = Y^m$  so that  $K \leq G$  as above, with  $T$  the unique minimal normal subgroup of  $Y$ . Then  $\text{soc}(K)$  is a normal subgroup of  $X$ , and a sub-direct subgroup of  $\text{soc}(G) = T^m$ .*

*Proof.* We first prove that  $\text{soc}(G) = T^m$ . Let  $H = T^m$ , and for each  $i \in M$  let  $T_i = \{(1, \dots, 1, t, 1, \dots, 1) \in G : t \in T\}$  where the non-trivial elements appear in the  $i^{\text{th}}$  entry. Thus we have  $T_i \leq H$  and  $H = \langle T_i : i \in M \rangle$ . Since  $T$  is a minimal normal subgroup of  $Y$  it follows that, for each  $i$ ,  $T_i$  is a minimal normal subgroup of  $G$ . Thus  $H \leq \text{soc}(G)$ . Now let  $R$  be any minimal normal subgroup of  $G$ . Since  $H$  and  $R$  are both normal in  $G$  it follows that  $R \cap H$  is normal in  $G$ , and since  $R$  is a minimal normal subgroup of  $G$  then either  $R \cap H = 1$  or  $R \cap H = R$ .

Suppose that  $R \cap H = 1$ . Then, by the second isomorphism theorem,  $R \cong RH/H \leq G/H$ . Schreier's Conjecture (which is known to be true by the classification of finite simple groups) states that  $\text{Aut}(T)/\text{Inn}(T) \cong \text{Aut}(T)/T$  is soluble. Thus it follows that  $Y/T$  is soluble, and therefore  $(Y/T)^m \cong Y^m/T^m = G/H$  is soluble. Consequently  $R$  is soluble. Since  $R \neq 1$  there exists  $i$  such that  $1 \neq \varphi_i(R) \leq Y$ . However, since  $R$  is soluble it follows that  $\varphi_i(R)$  is soluble which is a contradiction as  $Y$  is almost simple. Thus  $R \leq H$  and so  $H = \text{soc}(G)$ . We also note that following a similar argument proves that any non-abelian simple subgroup of  $G$  is necessarily a subgroup of  $H = \text{soc}(G)$ .

Now let  $R$  be a minimal normal subgroup of  $K$ . We claim that  $R \cong T$ . To prove this claim, we first consider the homomorphism  $\varphi_i : K \rightarrow S_q$  defined in (2.5). As both  $\ker(\varphi_i)$  and  $R$  are normal in  $K$ , and because  $R$  is minimal, it follows that  $R \cap \ker(\varphi_i) = 1$  or  $R$ . If  $R \cap \ker(\varphi_i) = R$  for all  $i$  then  $\varphi_i(R) = 1$  for all  $i$ , which implies that  $R = 1$ , a contradiction. So there exists  $j$  such that  $R \cap \ker(\varphi_j) = 1$ , which implies that  $\varphi_j(R) \cong R$ . Moreover,  $1 \neq \varphi_j(R) \trianglelefteq \varphi_j(K) = Y$  and so  $T \leq \varphi_j(R)$  since  $Y$  is almost simple. Thus  $\text{soc}(\varphi_j(R)) = T$  and so  $\text{soc}(R) \cong T$ . Since  $R$  is normal in  $K$ , and because  $\text{soc}(R)$  is characteristic in  $R$ , it follows that  $\text{soc}(R)$  is normal in  $K$ . Therefore, as  $R$  is minimal, we deduce that  $R = \text{soc}(R)$ . Thus  $R \cong T$ , and so  $R$  is non-abelian simple. Thus, from above, it holds that  $R \leq \text{soc}(G)$ . As this holds for every minimal normal subgroup of  $K$ , we deduce that  $\text{soc}(K) \leq \text{soc}(G)$ .

Now, we have shown that there exists  $j$  such that  $\varphi_j(R) \cong T$ . Thus, because  $R \leq \text{soc}(K) \leq \text{soc}(G) = T^m$ , we conclude that  $\varphi_j(\text{soc}(K)) \cong T$ . As  $\text{soc}(K)$  is a characteristic subgroup of  $K$ , and  $K$  is a normal subgroup of  $X$ , it follows that  $\text{soc}(K)$  is a normal subgroup of  $X$ . By Proposition 2.8,  $X$  acts transitively on  $M$ , so, by letting  $X$  act on  $\text{soc}(K)$  via conjugation, it follows that  $\varphi_i(\text{soc}(K)) \cong T$  for all  $i$ .  $\square$

As  $\text{soc}(K)$  is a subdirect subgroup of a direct product of non-abelian simple groups, we can apply Scott's Lemma [43, p.328]. In particular, Scott's Lemma implies that there exists a partition  $\mathcal{J} = \{J_1, \dots, J_\ell\}$  of  $M$  such that

$$\text{soc}(K) = D_1 \times \dots \times D_\ell$$

where each  $D_i$  is a full diagonal subgroup of  $\Pi_{j \in J_i} T$ . We call  $J_i$  the *support* of  $D_i$ . Let  $x = (t_1, \dots, t_m) \in D_i$ . Then  $t_i = 1$  for all  $i \in M \setminus \{J_i\}$ , so we introduce the following notation.

**Notation 5.3.** Let  $J = \{i_1, \dots, i_k\} \subseteq M$  and  $t_{i_1}, \dots, t_{i_k}$  be  $k$  permutations of  $S_q$ . Then we let  $[t_{i_1}, \dots, t_{i_k}]_J$  denote the group element in the base group of  $\text{Aut}(\Gamma)$  given by

$$[t_{i_1}, \dots, t_{i_k}]_J|_u = \begin{cases} t_u & \text{if } u \in J \\ 1 & \text{if } u \notin J \end{cases}$$

Now, because  $D_i$  is a full diagonal subgroup of  $\Pi_{j \in J_i} T$ , it follows that

$$D_i = \{[t, t^{\psi_{i_2}}, \dots, t^{\psi_{i_k}}]_{J_i} : t \in T\}$$

where each  $\psi_{i_s}$  is an automorphism of  $T$ . Thus  $D_i \cong T$  for each  $i$ , and so  $\text{soc}(K)$  is the direct product of a finite set of non-abelian simple groups. Hence,  $\mathcal{D} = \{D_1, \dots, D_\ell\}$  is the complete set of minimal normal subgroups of  $\text{soc}(K)$  [16, p.113]. By Proposition 5.2,  $\text{soc}(K)$  is normal in  $X$ , so  $X$  acts on  $\mathcal{D}$  by conjugation. In particular, for each  $D_i \in \mathcal{D}$  and  $x \in X$ , there exists  $D_u \in \mathcal{D}$  such that  $x^{-1}D_i x = D_u$ .

**Lemma 5.4.** Let  $x = h\sigma \in X$ . Then  $x^{-1}D_i x = D_j$  if and only if  $J_i^\sigma = J_j$ .

*Proof.* Let  $1 \neq y \in D_i$ , and suppose that  $J_i^\sigma = J_j$ . As  $y_s \neq 1$  for all  $s \in D_i$ , it follows that  $x^{-1}yx|_s \neq 1$  for all  $s \in J_j$ . Hence  $x^{-1}yx \in D_j$ , and so  $x^{-1}D_i x = D_j$  by the comments preceding this lemma. Conversely suppose that  $x^{-1}D_i x = D_j$ , and let  $s \in J_i$ , so  $y_s \neq 1$ . Then  $x^{-1}yx|_{s^\sigma} \neq 1$ , and because  $x^{-1}yx \in D_j$  it follows that  $s^\sigma \in J_j$ .  $\square$

**Remark 5.5.** As  $X$  acts on  $\mathcal{D}$  by conjugation, it is a consequence of Lemma 5.4 that  $\mathcal{J}$  is an  $X$ -invariant partition of  $M$ . Moreover, because  $X$  acts transitively on  $M$  it also follows that  $X$  acts transitively on  $\mathcal{D}$ .

Let us consider the group

$$D_i = \{[t, t^{\psi_{i_2}}, \dots, t^{\psi_{i_k}}]_{J_i} : t \in T\}$$

and the group  $\overline{N_{S_q}(T)} \leq \text{Aut}(T)$ . (Here  $\overline{N_{S_q}(T)}$  denotes that subgroup of  $\text{Aut}(T)$  induced by  $N_{S_q}(T)$ .) Let  $\mathcal{T}$  be a transversal for  $\overline{N_{S_q}(T)}$  in  $\text{Aut}(T)$ . Then for each  $s \in J_i$ ,  $\psi_{i_s} = z_{i_s} \bar{h}_{i_s}$  for some  $z_{i_s} \in \mathcal{T}$  and  $\bar{h}_{i_s} \in \overline{N_{S_q}(T)}$ . Conjugating  $D_i$  by  $[1, h_{i_2}^{-1}, \dots, h_{i_k}^{-1}]_{J_i}$  yields

$$D_i^{[1, h_{i_2}^{-1}, \dots, h_{i_k}^{-1}]_{J_i}} = \{[t, t^{z_{i_2}}, \dots, t^{z_{i_k}}]_{J_i} : t \in T\}.$$

Let  $y = \prod_{i=1}^{\ell} [1, h_{i_2}^{-1}, \dots, h_{i_k}^{-1}]_{J_i}$ . By replacing  $C$  by  $C^y$  if necessary, we may assume without loss of generality that for all  $D_i$ , each  $\psi_{i_s}$  lies in the transversal  $\mathcal{T}$ .

**Definition 5.6.** Let  $\mathcal{T}$  be a transversal for  $\overline{N_{S_q}(T)}$  in  $\text{Aut}(T)$  and  $D_i = \{[t, t^{\psi_{i_2}}, \dots, t^{\psi_{i_k}}]_{J_i} : t \in T\}$  where each  $\psi_{i_s} \in \mathcal{T}$ . If  $\psi_{i_s} = 1$  for all  $i_s \in J_i$  we say that  $D_i$  has Form 1, otherwise we say that  $D_i$  has Form 2.

Using the classification of finite almost simple 2-transitive groups we know that  $|\mathcal{T}| \leq 2$  [11, Table 7.4]. If  $|\mathcal{T}| = 1$  then  $D_i$  has to have 1. If  $D_i$  has 2 then  $T$  is the socle of one of the groups from the third column of Table 1 and  $\mathcal{T} = \{1, \tau\}$  for some  $\tau \in \text{Aut}(T) \setminus \overline{N_{S_q}(T)}$  [11, Table 7.4]. Moreover we can assume that  $\tau^2 = 1$  (see [23, Remark 4.1]). Recall that, by assumption, each  $\psi_{i_s}$  is equal to 1 or  $\tau$ .

**Lemma 5.7.** Every minimal normal subgroup of  $\text{soc}(K)$  has the same Form, i.e. all the  $D_i$  have 1, or they all have 2.

*Proof.* Suppose there exist  $D_i, D_j \in \mathcal{D}$  such that  $D_i$  has 1 and  $D_j$  has 2. Let  $J_i = \{i_1, \dots, i_k\}$ ,  $J_j = \{j_1, \dots, j_k\}$  be the respective supports of  $D_i, D_j$ . As  $D_j$  has 2, there exists  $s \in J_j$  such that  $\varphi_s = \tau$ . By Remark 5.5, there exists  $x = (h_1, \dots, h_m)\sigma \in X$  such that  $x^{-1}D_i x = D_j$ , and so  $J_i^\sigma = J_j$ . Let  $u, v \in J_i$  such that  $u^\sigma = j_1$  and  $v^\sigma = s$ . Then for each  $t \in T$  there exists  $t' \in T$  such that  $[t, \dots, t]_{J_i}^x = [t', t'^{\psi_{j_2}}, \dots, t'^{\psi_{j_k}}]_{J_j}$ . This implies that for each  $t \in T$  there exists  $t' \in T$  such that  $t^{h_u} = t'$  and  $t^{h_s} = t'^\tau$ , which in turn implies that  $t^{h_s} = t^{h_u \tau}$ . Hence, as automorphisms of  $T$ , we have that  $\bar{h}_s = \bar{h}_u \tau$ . However this implies that  $\tau \in \overline{N_{S_q}(T)}$  which is a contradiction.  $\square$

Suppose there exists  $D_i \in \mathcal{D}$ , with support  $J_i$ , that has 2. We define

$$(5.1) \quad J_i^{(1)} = \{s \in J_i : \psi_s = 1\} \text{ and } J_i^{(2)} = \{s \in J_i : \psi_s = \tau\}.$$

It follows that  $J_i$  is the disjoint union of  $J_i^{(1)}$  and  $J_i^{(2)}$ . Because  $\mathcal{J}$  is a partition of  $M$  and since  $S_m$  acts  $m$ -transitively on  $M$ , there exists  $\sigma \in \Pi \text{Sym}(J_i) \leq L \cong S_m$  such that

$$D_i^\sigma = \{[t, \dots, t, t^\tau, \dots, t^\tau]_{J_i} : t \in T\}$$

for each  $i$ . Therefore, replacing  $C$  by  $C^\sigma$ , we can assume that each  $D_i$  has this form. We want to be able to refer to the two possibilities for  $D_i$ . Therefore we define the following.

**Definition 5.8.** We say **Case 1** holds if 1 holds for all  $D_i$  (that is,  $\psi_s = 1$  for all  $s \in M$ ) and

$$D_i = \{[t, \dots, t]_{J_i} : t \in T\}$$

for  $1 \leq i \leq \ell$ . In this case we abbreviate  $[t, \dots, t]_{J_i}$  (as in Notation 5.3) by  $[t]_{J_i}$  where

$$[t]_{J_i}|_u = \begin{cases} t & \text{if } u \in J_i \\ 1 & \text{if } u \notin J_i. \end{cases}$$

We say **Case 2** holds if  $T$  is the socle of one of the groups from the third column of Table 1 and  $q$  is the corresponding degree;  $\tau \in \text{Aut}(T) \setminus \overline{N_{S_q}(T)}$  such that  $\tau^2 = 1$ ; 2 holds for all  $D_i$  (that is there exists  $s \in J_i$  such that  $\psi_s \neq 1$ ); and

$$D_i = \{[t, \dots, t, t^\tau, \dots, t^\tau]_{J_i} : t \in T\}$$

for  $1 \leq i \leq \ell$ . In this case we abbreviate  $[t, \dots, t, t^\tau, \dots, t^\tau]_{J_i}$  (as in Notation 5.3) by  $[t, t^\tau]_{J_i}$  where

$$[t, t^\tau]_{J_i}|_u = \begin{cases} t & \text{if } u \in J_i^{(1)} \\ t^\tau & \text{if } u \in J_i^{(2)} \\ 1 & \text{if } u \notin J_i. \end{cases}$$

**Lemma 5.9.** Suppose  $D_i \in \mathcal{D}$ , with support  $J_i$ , has 2. Then  $\{J_i^{(1)}, J_i^{(2)}\}$  is an  $X_{J_i}$ -invariant partition of  $J_i$ , and in particular  $|J_i^{(1)}| = |J_i^{(2)}|$ .

*Proof.* Let  $x = h\sigma = (h_1, \dots, h_m)\sigma \in X_{J_i}$  and suppose  $\emptyset \neq (J_i^{(1)})^\sigma \cap J_i^{(1)} \subset J_i^{(1)}$ , that is, properly contained in  $J_i^{(1)}$ . Then there exist  $u, s \in J_i^{(1)}$  such that  $u^\sigma \in J_i^{(1)}$  and  $s^\sigma \in J_i^{(2)}$ . Following a similar argument to that used in the proof of Lemma 5.7, we deduce that, as automorphisms of  $T$ ,  $\bar{h}_s = \bar{h}_u\tau$ . This implies that  $\tau \in \overline{N_{S_q}(T)}$ , which is a contradiction. Thus either  $(J_i^{(1)})^\sigma = J_i^{(1)}$  or  $J_i^{(1)} \cap J_i^{(1)} = \emptyset$ . A similar argument shows this to be true for  $J_i^{(2)}$  also. The result now follows from the fact that  $X_{J_i}$  acts transitively on  $J_i$ .  $\square$

**Remark 5.10.** (i) It is a consequence of Lemma 5.9 that  $\mathcal{J}^{(\tau)} = \{J_1^{(1)}, J_1^{(2)}, \dots, J_\ell^{(1)}, J_\ell^{(2)}\}$  is an  $X$ -invariant partition of  $M$ .

(ii) Let  $|\mathcal{D}| = \ell$ . As  $X$  acts transitively on  $M$ ,  $|J_i| = m/\ell = k$  for all  $i$ , in particular,  $m = \ell k$ . If **Case 2** holds, Lemma 5.9 implies that  $k$  is even. Let  $\alpha = (\alpha_1, \dots, \alpha_m) \in C$ ,  $J \in \mathcal{J}$  and  $i \in J$  if **Case 1** holds, and  $i \in J^{(1)}$  if **Case 2** holds. Since  $T$  acts transitively on  $Q$ , there exists  $t \in T$  such that  $\alpha_i^t \neq \alpha_i$ . Let  $x = [t]_J$  in **Case 1** and  $x = [t, t^\tau]_J$  in **Case 2**, so  $x \in \text{soc}(K)$ . As  $\alpha_i \neq \alpha_i^t$ , it follows that  $\alpha \neq \alpha^x$ , so  $3 \leq \delta \leq d(\alpha, \alpha^x)$ . Hence  $k \geq 3$ , and if **Case 2** holds,  $k \geq 4$  as  $k$  is even.

## 6. THE STRUCTURE OF THE PROJECTION CODES

In the previous section we proved that there exists an  $X$ -invariant partition  $\mathcal{J}$  of  $M$  for any alphabet almost simple  $X$ -neighbour transitive code with  $\delta \geq 3$  and  $K := X \cap B \neq 1$ . Moreover, if **Case 2** holds (as in Definition 5.8) Lemma 5.9 implies that  $\mathcal{J}^{(\tau)}$  is also an  $X$ -invariant partition of  $M$ . Let  $J \in \mathcal{J}$  if **Case 1** holds and  $J \in \mathcal{J}^{(\tau)}$  if **Case 2** holds. Then Lemma 3.4 implies that  $\pi_J(C)$  is a  $\chi(X_J)$ -neighbour transitive code in  $H(J, q)$ . In this section we prove Proposition 6.1, which describes the code  $\pi_J(C)$  in

each case. Recall from Example 2.2 the repetition code  $\text{Rep}(k, q)$  in  $H(k, q)$ , and from Example 4.3 the code  $\text{All}(pq, q)$  in  $H(pq, q)$ .

**Proposition 6.1.** *Let  $C$  be an alphabet almost simple  $X$ -neighbour transitive code with  $\delta \geq 3$  and  $K \neq 1$ . Then if Case 1 holds, either*

- (i)  $\pi_J(C) = \text{Rep}(k, q)$  in  $H(J, q)$  with  $|J| = k$  for all  $J \in \mathcal{J}$ , or
- (ii)  $\pi_J(C) \subseteq \text{All}(pq, q)$  in  $H(J, q)$  with  $|J| = pq$  for some positive integer  $p$  for all  $J \in \mathcal{J}$ .

If Case 2 holds then  $\pi_J(C) \subseteq \text{All}(pq, q)$  in  $H(J, q)$  with  $|J| = pq$  for some positive integer  $p$  for all  $J \in \mathcal{J}^{(\tau)}$ .

*Proof.* Let  $J \in \mathcal{J}$  if Case 1 holds and  $J \in \mathcal{J}^{(\tau)}$  if Case 2 holds. Also let  $|J| = r$  and denote  $H(J, q)$  by  $\Gamma(J)$ . As  $\text{soc}(K)$  is contained in the base group of  $\text{Aut}(\Gamma)$ , it follows that  $\text{soc}(K) \trianglelefteq X_J$ . Hence  $\chi(\text{soc}(K)) \trianglelefteq \chi(X_J)$  in  $\text{Aut}(\Gamma(J))$ . Thus  $\chi(X_J) \leq N := N_{\text{Aut}(\Gamma(J))}(\chi(\text{soc}(K)))$ . We claim that

$$(6.1) \quad N = \text{Diag}_r(N_{S_q}(T)) \rtimes S_r.$$

By Remark 3.2,  $\chi(\text{soc}(K)) = \text{Diag}_r(T)$ , and it is clear that the top group  $L(J) \cong S_r$  of  $\text{Aut}(\Gamma(J)) \cong S_q \text{ wr } S_r$  centralises  $\chi(\text{soc}(K))$ , so  $L(J) \leq N$ . Hence if  $h\sigma \in N$  it follows that  $h = (h_{i_1}, \dots, h_{i_r}) \in N$ . Now, for  $t \in T$  it follows that

$$(t, \dots, t)^h = (t^{h_{i_1}}, \dots, t^{h_{i_r}}) \in \chi(\text{soc}(K)).$$

Thus, for all  $t \in T$  and  $j \geq 2$ ,  $t^{h_{i_1}} = t^{h_{i_j}}$ . This implies that  $h_{i_1}h_{i_j}^{-1} \in C_{S_q}(T)$ . However, because  $T$  is almost simple and acts primitively on  $Q$ ,  $C_{S_q}(T) = 1$  [16, Theorem 4.2A]. As  $h \in N$  it follows that for each  $t \in T$  there exists  $t' \in T$  such that  $(t, \dots, t)^h = (t', \dots, t')$ . In particular, this implies that  $h_{i_1} \in N_{S_q}(T)$ . Thus  $N \leq \text{Diag}_r(N_{S_q}(T)) \rtimes S_r$ , and it is straight forward to show that these two groups are in fact equal. Hence the claim holds.

Now, because  $J$  is a block of imprimitivity for the action of  $X$  on  $M$ , Lemma 3.4 implies that  $\pi_J(C)$  is a  $\chi(X_J)$ -neighbour transitive code in  $\Gamma(J)$ . As  $\chi(X_J) \leq \text{Diag}_r(N_{S_q}(T)) \rtimes S_r$ , we conclude that  $\pi_J(C)$  is a diagonally neighbour transitive code in  $\Gamma(J)$ . Hence, by the classification of diagonally neighbour transitive codes [20], one of the following holds:

- (a)  $\pi_J(C)$  is the repetition code  $\text{Rep}(r, q)$
- (b)  $r < q$  and  $\pi_J(C) = \{(\alpha_{i_1}, \dots, \alpha_{i_r}) : \alpha_u \neq \alpha_v \text{ for all } u, v \in J\}$ ,
- (c)  $r$  is odd,  $q = 2$  and  $\pi_J(C) = \{\alpha \in \Gamma(J) : \alpha \text{ has weight } \frac{r-1}{2} \text{ or } \frac{r+1}{2}\}$ , or
- (d) there exists an integer  $p$  such that  $r = pq$  and  $\pi_J(C) \subseteq \text{All}(pq, q)$  in  $\Gamma(J)$ .

Let  $\delta_J$  be the minimum distance of  $\pi_J(C)$ . In each of the cases (a)–(d),  $\pi_J(C)$  is not the complete code in  $\Gamma(J)$ . Therefore Corollary 3.7 implies that  $\delta_J \geq 2$ . If (b) holds then we saw in Example 4.1 that  $\delta_J = 1$ , which is a contradiction. Suppose that (c) holds. Then  $q = 2$  and  $T \leq S_2$ . However,  $S_2$  is not almost simple, which is a contradiction. Therefore either (a) or (d) holds for  $\pi_J(C)$ . We note that if  $\pi_J(C) = \text{Rep}(r, q)$  then  $\delta_J = r$ . We claim that if  $\pi_J(C) \subseteq \text{All}(pq, q)$  then  $2 \leq \delta_J < r$ .

When (d) holds, the parameter  $r$  is equal to  $pq$  for some positive integer  $p$ . Also, by Remark 3.2,  $\chi(\text{soc}(K)) = \text{Diag}_r(T)$ . Since  $\pi_J(C)$  is contained in  $\text{All}(pq, q)$ , which has minimum distance 2, it follows that  $\delta_J \geq 2$ . Now suppose that  $\delta_J = r$ . Then as  $\pi_J(C)$  is neighbour transitive, Lemma 2.4 implies that  $\pi_J(C)$  is equivalent to  $\text{Rep}(r, q)$ . In particular  $|\pi_J(C)| = q$ . Let  $\alpha \in \pi_J(C)$  and suppose there exist  $t_1, t_2 \in T$  such that

$$\alpha^{(t_1, \dots, t_1)} = \alpha^{(t_2, \dots, t_2)}.$$

Then, because every letter of  $Q$  appears in  $\alpha$ , it follows that  $a^{t_1} = a^{t_2}$  for all  $a \in Q$ . That is  $t_1 = t_2$ . Hence  $\chi(\text{soc}(K))$  acts regularly on its orbits in  $\pi_J(C)$ . Therefore  $|T| \leq |\pi_J(C)| = q$ . In particular, as  $T$  acts transitively on  $Q$ , this implies that  $|T| = q$  and that  $T$  acts regularly on  $Q$ . Thus, by [16, Theorem 4.2A],  $C_{S_q}(T)$  acts transitively on  $Q$ , contradicting the fact that  $C_{S_q}(T) = 1$ . Hence  $\delta_J < r$  and the claim is proved. Therefore, to recap, we have shown that  $\delta_J \geq 2$ , and either (a) holds, or (d) holds with  $\delta_J < r$ .

Now let  $J_i, J_j \in \mathcal{J}$  and consider the codes  $\pi_{J_i}(C)$  and  $\pi_{J_j}(C)$  with minimum distances  $\delta_{J_i}$  and  $\delta_{J_j}$ . By Lemma 3.6, it follows that  $\delta_{J_i} = \delta_{J_j}$ . Thus, because the code in (a) has minimum distance  $\delta_J = r$  and the code in (d) has minimum distance  $2 \leq \delta_J < r$ , we conclude that if **Case 1** holds then either (i) or (ii) in the statement holds.

Assume now that **Case 2** holds, and recall from Definition 5.8 that  $T$  is the socle of one of the groups from the third column of Table 1. Consider  $J \in \mathcal{J}$ , so  $J^{(1)}, J^{(2)} \in \mathcal{J}^{(\tau)}$ . If we project onto  $J$ , we saw in the previous section that

$$\chi(\text{soc}(K)) = \{(\underbrace{t, \dots, t}_{k/2}, \underbrace{t^\tau, \dots, t^\tau}_{k/2}) \mid t \in T\}.$$

By Lemma 5.9,  $\{J^{(1)}, J^{(2)}\}$  is a  $\chi(X_J)$ -invariant partition of  $J$ . Moreover, by (6.1) we have that  $\chi(X_{J^{(1)}})$  and  $\chi(X_{J^{(2)}})$  are subgroups of  $\text{Diag}_{k/2}(N_{S_q}(T)) \rtimes S_{k/2}$ . Hence, for each  $x \in \chi(X_J)$  there exist  $h_1, h_2 \in N_{S_q}(T)$  and  $\sigma \in S_{k/2} \text{ wr } S_2$  such that

$$x = (\underbrace{h_1, \dots, h_1}_{k/2}, \underbrace{h_2, \dots, h_2}_{k/2})\sigma.$$

Thus for each  $t \in T$ , it follows that

$$x^{-1}(t, \dots, t, t^\tau, \dots, t^\tau)x = \begin{cases} (t^{\bar{h}_1}, \dots, t^{\bar{h}_1}, t^{\tau\bar{h}_2}, \dots, t^{\tau\bar{h}_2}) & \text{if } \sigma \text{ stabilises } J^{(1)} \\ (t^{\tau\bar{h}_2}, \dots, t^{\tau\bar{h}_2}, t^{\bar{h}_1}, \dots, t^{\bar{h}_1}) & \text{otherwise.} \end{cases}$$

As  $\chi(\text{soc}(K)) \trianglelefteq \chi(X_J)$ , we deduce in both cases that  $t^{\bar{h}_2} = t^{\tau\bar{h}_1\tau}$  for all  $t \in T$  (recall that  $\tau$  was chosen so  $\tau^2 = 1$ ). Now, because  $\overline{N_{S_q}(T)} \cong N_{S_q}(T)$  is a normal subgroup of  $\text{Aut}(T)$  for each of the possible groups  $T$ , one can deduce that  $\bar{h}_2 = \tau\bar{h}_1\tau = \bar{h}_1^\tau$ . This implies that  $h_2h_1^{-\tau} \in C_{S_q}(T) = 1$ , so  $h_2 = h_1^\tau$ . Thus for each  $x \in \chi(X_J)$  there exist  $h \in N_{S_q}(T)$  and  $\sigma \in S_{k/2} \text{ wr } S_2$  such that

$$x = (h, \dots, h, h^\tau, \dots, h^\tau)\sigma.$$

Now suppose that either  $\pi_{J^{(1)}}(C)$  or  $\pi_{J^{(2)}}(C)$  is the repetition code. By considering their minimum distances and applying a similar argument to the one above, we deduce that both codes are the repetition

code in their respective Hamming graphs. Since  $\mathcal{J}$  is an  $X$ -invariant partition of  $M$ , Proposition 3.4 implies that  $\pi_J(C)$  is also a  $\chi(X_J)$ -neighbour transitive code. As both  $\pi_{J(1)}(C)$  and  $\pi_{J(2)}(C)$  are the repetition codes, we deduce that every codeword of  $\pi_J(C)$  is of the form

$$\underbrace{(a, \dots, a)}_{k/2}, \underbrace{(b, \dots, b)}_{k/2}$$

for some  $a, b \in Q$ . Thus  $\delta_J = k/2$  or  $k$ . Let  $\alpha = (a, \dots, a, b, \dots, b) \in \pi_J(C)$ . For each possible group  $T$ ,  $(N_{S_q}(T)_a)^\tau$  has two orbits on  $Q$ , each of length at least 2. (Here  $(N_{S_q}(T)_a)^\tau$  denotes the stabiliser of  $a$  in  $N_{S_q}(T)$  under the automorphism  $\tau$ .) Let  $c, d \in Q \setminus \{b\}$  such that  $b, c$  are in the same  $(N_{S_q}(T)_a)^\tau$ -orbit and  $b, d$  are not in the same  $(N_{S_q}(T)_a)^\tau$ -orbit. Now let  $\nu_1 = \nu(\alpha, k/2 + 1, c)$  and  $\nu_2 = \nu(\alpha, k/2 + 1, d)$ , which are both neighbours of  $\alpha$ . Since  $\pi_J(C)$  is  $\chi(X_J)$ -neighbour transitive, there exists  $x = (h, \dots, h, h^\tau, \dots, h^\tau)\sigma \in \chi(X_J)$  such that  $\nu_1^x = \nu_2$ . Suppose  $(J^{(1)})^\sigma = J^{(2)}$ . Then, as  $\nu_1^x = \nu_2$ , it follows that  $a^h = b$  and  $a^h = d$ , which is a contradiction given that  $b \neq d$ . Therefore  $(J^{(1)})^\sigma = J^{(1)}$ , and so  $a^h = a$  and  $h^\tau \in (N_{S_q}(T)_a)^\tau$ . However, it then follows that either  $c^{h^\tau} = d$  or  $b^{h^\tau} = d$ , contradicting the choice of  $b, c$  and  $d$ . Hence we conclude that neither  $\pi_{J(1)}(C)$  nor  $\pi_{J(2)}(C)$  is the repetition code. Thus (d) holds for both  $\pi_{J(1)}(C)$  and  $\pi_{J(2)}(C)$ . It follows from this argument that  $\pi_{J^*}(C)$  cannot be the repetition code for any  $J^* \in \mathcal{J}^{(\tau)}$ , that is,  $\pi_{J^*}(C) \subseteq \text{All}(pq, q)$  for all  $J^* \in \mathcal{J}^{(\tau)}$ .  $\square$

## 7. BUILDING BLOCKS OF $C$

Let  $C$  be an alphabet almost simple  $X$ -neighbour transitive code with  $\delta \geq 3$  and  $K := X \cap B \neq 1$ , and recall from Section 5 that there exists an  $X$ -invariant partition  $\mathcal{J}$  of  $M$ . Let  $\hat{C}$  denote the projection code  $\pi_J(C)$  for some  $J \in \mathcal{J}$ , and let  $k = |J|$ . Also let  $\hat{S} = \chi(\text{soc}(K))$ . In this section we describe certain  $\hat{S}$ -orbits in  $\hat{C}$ . We then use these to describe a  $\text{soc}(K)$ -orbit in  $C$ .

7.1. Assume that **Case 1** holds, so

$$\hat{S} = \{x_t = (t, \dots, t) : t \in T\}.$$

Let  $\alpha \in \hat{C}$  and define

$$\hat{\Delta} = \alpha^{\hat{S}},$$

the  $\hat{S}$ -orbit containing  $\alpha$ . By Proposition 6.1, either  $\hat{C}$  is the repetition code or  $\hat{C} \subseteq \text{All}(pq, q)$  where  $p = k/q$  is a positive integer. Suppose that the former holds. Then there exists  $a \in Q$  such that  $\alpha = (a, \dots, a)$ . For  $t \in T$  it follows that

$$\alpha^{x_t} = (a, \dots, a)^{x_t} = (a^t, \dots, a^t).$$

Because  $T$  is acting transitively on  $Q$ , we deduce that  $\hat{\Delta} = \hat{C}$ .

Now suppose that  $\hat{C} \subseteq \text{All}(pq, q)$  with  $p = k/q$ , and let us identify  $Q$  with the set  $\{1, \dots, q\}$ . As every letter of  $Q$  appears  $p$  times in  $\alpha$ , there exists  $\sigma$  in the top group of  $\text{Aut}(H(k, q))$  such that

$$\alpha^\sigma = (1, 2, \dots, q, \dots, 1, \dots, q) = (\alpha(1), \dots, \alpha(1)) = \text{rep}_p(\alpha(1)).$$



By replacing  $\hat{C}$  with  $\hat{C}^\sigma$  if necessary, we can assume that  $\alpha = \text{rep}_p(\alpha(1)) \in \hat{C}$ . Note that as  $\hat{S}$  is centralised by the top group of  $\text{Aut}(H(k, q))$ ,  $\hat{S}$  is left unchanged by doing this. Now let  $x_t \in \hat{S}$ . It follows that

$$(7.1) \quad \text{rep}_p(\alpha(1))^{x_t} = (1, \dots, q, \dots, 1, \dots, q)^{x_t} = (1^t, \dots, q^t, \dots, 1^t, \dots, q^t) = \text{rep}_p(\alpha(t)).$$

As  $\hat{S} = \{x_t : t \in T\}$ , we deduce that

$$\hat{\Delta} = \alpha^{\hat{S}} = \text{Rep}_p(C(T)).$$

7.2. Suppose that Case 2 holds and let  $\alpha \in \hat{C}$ . We saw in Remark 5.10 that  $k$  is even, and by Proposition 6.1,  $\pi_{J^{(i)}}(C) \subseteq \text{All}(pq, q)$  with  $p = k/2q$  for  $i = 1, 2$ . Thus every letter in  $Q$  appears in  $\alpha$  exactly  $p$  times on  $J^{(1)}$ , and similarly for  $J^{(2)}$ . Consequently there exists  $\sigma$  in the top group of  $\text{Aut}(H(k, q))$  that fixes  $J^{(1)}$  setwise such that

$$(7.2) \quad \alpha^\sigma = (\alpha(1), \dots, \alpha(1))$$

which consists of  $2p$  repeated copies of  $\alpha(1)$ . As before, by replacing  $\hat{C}$  if necessary, we can assume that  $\alpha$  is as in (7.2). Now, in this case

$$\hat{S} = \{(\underbrace{t, \dots, t}_{pq}, \underbrace{t^\tau, \dots, t^\tau}_{pq}) : t \in T\}.$$

In order to give a nice description of the  $\hat{S}$ -orbit of  $\alpha$ , we again conjugate  $\hat{C}$  by an element in the top group so that  $\hat{S}$  looks slightly different. Let

$$\sigma' = \Pi_{i=1}^p \Pi_{j=1}^q ((2i-1)q + j, (2i-1)pq + j).$$

and replace  $\hat{C}$  by  $\hat{C}^{\sigma'}$ . Note that  $\alpha^{\sigma'} = \alpha$  and

$$(7.3) \quad \hat{S}^{\sigma'} = \{x_t^{(\tau)} = (\underbrace{t, \dots, t}_q, \underbrace{t^\tau, \dots, t^\tau}_q, \dots, \underbrace{t, \dots, t}_q, \underbrace{t^\tau, \dots, t^\tau}_q) : t \in T\},$$

so that after replacement we can assume that  $\alpha$  is as in (7.2) and that  $\hat{S}$  has the form above. We can identify  $\alpha$  with the vertex  $\text{rep}_p(\alpha(1, 1^\tau))$  where  $\alpha(t, t^\tau) = (\alpha(t), \alpha(t^\tau))$  for any  $t \in T$ . Once we've made this identification, it follows from a direct calculation similar to (7.1) that for  $t \in T$ ,

$$\alpha^{x_t^{(\tau)}} = \text{rep}_p(\alpha(1, 1^\tau))^{x_t^{(\tau)}} = \text{rep}_p(\alpha(t, t^\tau)).$$

Hence we deduce that

$$\hat{\Delta} = \alpha^{\hat{S}} = \text{Rep}_p(C(T, T^\tau)).$$

7.3. **Piecing the parts back together.** Let  $\alpha \in C$  and

$$\Delta = \alpha^{\text{soc}(K)}.$$

As  $\text{soc}(K)$  is equal to the direct product of the groups  $D_i \in \mathcal{D}$ , it follows that we can identify  $\Delta$  with the Cartesian product of the  $D_i$ -orbits on  $\alpha$ . That is

$$\Delta = \alpha^{D_1} \times \dots \times \alpha^{D_\ell}$$

Because each  $D_i$  has support  $J_i$ , it follows that  $D_i$  leaves  $\alpha$  unchanged on the set of entries  $M \setminus J_i$ . So we can identify  $\alpha^{D_i}$  with  $\pi_{J_i}(\alpha^{\chi(D_i)}) = \pi_{J_i}(\alpha)^{\chi(D_i)}$ . (The idea here is that we are throwing away the part of  $\alpha$  that is left unchanged when  $D_i$  acts on it.) We also note that  $\chi(D_i) = \chi(\text{soc}(K))$  when we project onto  $J_i$ . Hence, in each case, by replacing  $C$  with an equivalent code if necessary, we can identify  $\Delta$  with the Cartesian product of the orbit  $\hat{\Delta}$  described in Section 7.1 or Section 7.2.

More specifically, suppose that **Case 1** holds with  $\hat{\Delta} = \pi_J(C) = \text{Rep}(k, q)$  for all  $J \in \mathcal{J}$ . Then  $\Delta$  is equal to the Cartesian product of  $\ell$ -copies of the repetition code. This is just the product construction applied to  $\text{Rep}(k, q)$ , as defined in Section 3.1, so

$$\Delta = \text{Prod}_\ell(\text{Rep}(k, q)).$$

Let  $\beta \in C$ . As  $\pi_J(\beta) \in \text{Rep}(k, q)$  for all  $J \in \mathcal{J}$ , there exist  $a_1, \dots, a_\ell \in Q$  such that  $\beta = (a_1^k, \dots, a_\ell^k)$ . In particular,  $\beta \in \Delta$ , so in this case

$$(7.4) \quad C = \Delta = \text{Prod}_\ell(\text{Rep}(k, q)).$$

Suppose now that **Case 1** holds such that for all  $J \in \mathcal{J}$ ,  $\pi_J(C) \subseteq \text{All}(pq, q)$  and  $k = pq$  for some positive integer  $p$ . Then there exists  $\sigma \in \Pi_{i=1}^\ell \text{Sym}(J_i)$  that centralises  $\text{soc}(K)$  such that  $\alpha^\sigma = (\alpha(1), \dots, \alpha(1))$ . Hence, by replacing  $C$  by  $C^\sigma$  if necessary, it follows that  $\Delta$  is the Cartesian product of  $\hat{\Delta} = \text{Rep}_p(C(T))$ . This is just the product construction applied to  $\text{Rep}_p(C(T))$ , that is,

$$(7.5) \quad \Delta = \text{Prod}_\ell(\text{Rep}_p(C(T))).$$

If **Case 2** holds then we choose  $\sigma \in \Pi_{i=1}^\ell \text{Sym}(J_i)$  so that  $\alpha^\sigma = (\alpha(1), \dots, \alpha(1))$  but also so that each  $\chi(D_i)$  is as in (7.3). By replacing  $C$  with  $C^\sigma$ , it follows that

$$(7.6) \quad \Delta = \text{Prod}_\ell(\text{Rep}_p(C(T, T^\tau))).$$

## 8. PROOF OF THEOREM 1.1

Let  $C$  be an alphabet almost simple  $X$ -neighbour transitive code with  $\delta \geq 3$  and  $K := X \cap B \neq 1$  in  $H(m, q)$ . Then  $X_1^Q$  is a 2-transitive group of almost simple type. Let  $T$  be the minimal normal subgroup of  $X_1^Q$ , which is non-abelian simple. By replacing  $C$  by an equivalent code if necessary, it follows from Proposition 5.2 that  $\text{soc}(K)$  is a subdirect product of  $T^m$ . Thus, applying Scott's Lemma [43], there exists a partition  $\mathcal{J} = \{J_1, \dots, J_\ell\}$  of  $M$  such that

$$\text{soc}(K) = D_1 \times \dots \times D_\ell$$

where each  $D_i$  is a full diagonal subgroup of  $\Pi_{j \in J_i} T$ , and by Lemma 5.4,  $\mathcal{J}$  is an  $X$ -invariant partition of  $M$ . By again replacing  $C$  if necessary, it follows from Lemma 5.7 that all the subgroups  $D_i$  have the same Form, as in Definition 5.6, and that either **Case 1** or **Case 2** holds, as in Definition 5.8. Moreover, if **Case 2** holds, we deduce from Lemma 5.9 that there exists an  $X$ -invariant partition  $\mathcal{J}^{(\tau)}$  of  $M$  which is a refinement of the partition  $\mathcal{J}$ .

Now let  $\alpha \in C$ ,  $\Delta = \alpha^{\text{soc}(K)}$  and  $k = m/\ell$ . We deduce from Section 7.3 that, up to equivalence, either

$$\Delta = \begin{cases} \text{Prod}_\ell(\text{Rep}(k, q)) & \text{if Case 1 and Proposition 6.1-(i) hold,} \\ \text{Prod}_\ell(\text{Rep}_p(C(T))) \text{ with } k = pq & \text{if Case 1 and Proposition 6.1-(ii) hold, or} \\ \text{Prod}_\ell(\text{Rep}_p(C(T, T^\tau))) \text{ with } k = 2pq & \text{if Case 2 holds.} \end{cases}$$

As  $\Delta$  is a  $\text{soc}(K)$ -orbit, and because  $\text{soc}(K)$  is a normal subgroup of  $X$ ,  $\Delta$  is a block of imprimitivity for the action of  $X$  on  $C$ . Thus either  $C = \Delta$  (which is necessarily true if Case 1 and Proposition 6.1-(i) hold), or  $C$  is the disjoint union of  $X$ -translates of  $\Delta$ . Moreover, Lemma 2.7 implies that  $\Delta$  is neighbour transitive, which we claim is true in each case. In Example 2.2, we saw that  $\text{Rep}(k, q)$  is neighbour transitive, and in [20], the authors proved that for a subgroup  $H \leq S_q$ , the code  $C(H)$  is neighbour transitive if and only if  $N_{S_q}(H)$  is 2-transitive. Moreover, in joint work with Spiga [23], the authors proved that for each  $T$  in Table 1 and outer automorphism  $\tau$  with order 2, the code  $C(T, T^\tau)$  is neighbour transitive. Thus it follows from Lemma 3.1 and [20, Lemma 5] that in each case,  $\Delta$  is indeed a neighbour transitive code, proving the claim. Finally, we observe that if Case 1 and Proposition 6.1-(ii) hold or Case 2 holds, then  $\Delta$  is a frequency permutation array with each letter appearing  $\ell p$  or  $2\ell p$  times respectively.

## 9. ANOTHER EXAMPLE

In this section we demonstrate that for some of the codes  $C$  in Theorem 1.1, the projected codes  $\pi_J(C)$  may have minimum distance smaller than that of  $C$ , and indeed, may have minimum distance 2. We give an example of an alphabet almost simple  $X$ -neighbour transitive code with  $\delta = 3$  and  $X$ -invariant partition  $\mathcal{J}$  of  $M$  such that  $\pi_J(C)$  has minimum distance  $\delta(\pi_J(C)) = 2$  for each  $J \in \mathcal{J}$ .

**Example 9.1.** Let  $Q = \{1, \dots, q\}$  for some  $q \geq 5$  and define

$$C = \{(\alpha(t_1), \dots, \alpha(t_\ell)) \in H(\ell q, q) : t_i t_j^{-1} \in A_q \ \forall i, j\}.$$

Let  $R = \text{Diag}_q(S_q) \rtimes S_q \leq \text{Aut}(H(q, q))$  and

$$X = \{(x_{h_1} \sigma_1, \dots, x_{h_\ell} \sigma_\ell) \sigma \in R \text{ wr } S_\ell : h_i h_j^{-1} \in A_q, \sigma_i \sigma_j^{-1} \in A_q \text{ for all } i, j\}.$$

We claim that  $C$  is  $X$ -neighbour transitive.

*Proof.* Let  $\beta = (\alpha(t_1), \dots, \alpha(t_\ell)) \in C$  and  $x = (x_{h_1} \sigma_1, \dots, x_{h_\ell} \sigma_\ell) \sigma \in X$ . It follows from (3.1) and [20, Lemma 8] that

$$\begin{aligned} \beta^x &= (\alpha(t_1)^{x_{h_1} \sigma_1}, \dots, \alpha(t_\ell)^{x_{h_\ell} \sigma_\ell})^\sigma \\ &= (\alpha(\sigma_1^{-1} t_1 h_1), \dots, \alpha(\sigma_\ell^{-1} t_\ell h_\ell))^\sigma \\ &= (\alpha(\sigma_{1\sigma^{-1}}^{-1} t_{1\sigma^{-1}} h_{1\sigma^{-1}}), \dots, \alpha(\sigma_{\ell\sigma^{-1}}^{-1} t_{\ell\sigma^{-1}} h_{\ell\sigma^{-1}})). \end{aligned}$$

From the definition of  $C$  and  $X$ , we deduce that for all  $i, j$ ,

$$\sigma_i^{-1} t_i h_i (\sigma_j^{-1} t_j h_j)^{-1} = \sigma_i^{-1} t_i h_i h_j^{-1} t_i^{-1} t_j t_i^{-1} \sigma_i \sigma_j^{-1} \sigma_j \in A_q.$$

Therefore  $X \leq \text{Aut}(C)$ . Now let

$$\alpha = (\alpha(1), \dots, \alpha(1)) \in C.$$

Then  $y = (x_{t_1}, \dots, x_{t_\ell}) \in X$  and  $\alpha^y = \beta$ . Since  $\beta$  was arbitrarily chosen, it follows that  $X$  acts transitively on  $C$ .

To prove that  $X$  acts transitively on the neighbour set of  $C$ , we first describe the neighbours of  $\alpha$ . The neighbours of  $\alpha(1)$  in  $H(q, q)$  are

$$\Gamma_1(\alpha(1)) = \{\nu(\alpha(1), a, b) : a, b \in Q \text{ and } a \neq b\}.$$

Thus following the notation of Section 3.1, the neighbours of  $\alpha$  in  $H(\ell q, q)$  are

$$\Gamma_1(\alpha) = \{\gamma(\alpha, i, \nu(\alpha(1), a, b)) : i \in \{1, \dots, \ell\}, a, b \in Q \text{ and } a \neq b\}.$$

Consider the group

$$W = \{(x_{h_1}\sigma_1, \dots, x_{h_\ell}\sigma_\ell)\sigma \in X : h_i = \sigma_i \forall i\}.$$

Then  $W \leq X_\alpha$ . Let

$$\nu_1 = \gamma(\alpha, i, \nu(\alpha(1), a, b)), \nu_2 = \gamma(\alpha, j, \nu(\alpha(1), s, u))$$

be two neighbours of  $\alpha$ . Since  $A_q$  acts 2-transitively on  $Q$ , there exists  $h \in A_q$  such that  $a^h = s$  and  $b^h = u$ . Let  $x = (x_h\sigma, \dots, x_h\sigma) \in W$ . Since  $\sigma = h$ , we deduce from Lemma 2.4 and (3.1) that  $\nu_1^x = \gamma(\alpha, i, \nu(\alpha(1), s, u))$ . It follows that there exists  $\sigma' \in S_\ell$  such that  $i^{\sigma'} = j$  and so  $\nu_1^{x\sigma'} = \nu_2$ . Thus  $X_\alpha$  acts transitively on  $\Gamma_1(\alpha)$ , and so, because  $X$  acts transitively on  $C$ , we deduce that  $X$  acts transitively on the neighbour set of  $C$ . Hence  $C$  is  $X$ -neighbour transitive.  $\square$

Since  $C \subseteq \text{Prod}_\ell(C(S_q))$  and  $\delta(C(S_q)) = 2$ , it follows that  $C$  has minimum distance  $\delta \geq 2$ . If  $\delta = 2$ , then because  $X$  acts transitively on  $C$ , there exists  $\beta \in C$  such that  $d(\alpha, \beta) = 2$ . However, this holds if and only if  $\beta = (\alpha(1), \dots, \alpha(t_i), \dots, \alpha(1))$  for some transposition  $t_i \in S_q$ , and such a vertex is not a codeword. Thus  $\delta \geq 3$ . Let  $t_1$  be a 3-cycle in  $A_q$  and  $\beta = (\alpha(t_1), \alpha(1), \dots, \alpha(1)) \in C$ . Then  $d(\alpha, \beta) = 3$ . Hence  $C$  has minimum distance  $\delta = 3$ . Now, it is clear that  $\mathcal{J} = \{J_1, \dots, J_\ell\}$ , where  $J_i = \{a + (i-1)q : a \in Q\}$ , is an  $X$ -invariant partition of  $M$ . Because  $(\alpha(t), \dots, \alpha(t)) \in C$  for all  $t \in S_q$ , we have that  $\pi_J(C) = C(S_q)$  for all  $J \in \mathcal{J}$ . Thus  $C$  is an example of an  $X$ -neighbour transitive code with minimum distance 3 and  $X$ -invariant partition  $\mathcal{J}$  such that  $\pi_J(C)$  has minimum distance 2 for each  $J \in \mathcal{J}$ .

We observe that  $K = X \cap B = \{(x_{h_1}, \dots, x_{h_\ell}) : h_i h_j^{-1} \in A_q\}$ . Also,  $(x_h, \dots, x_h) \in K$  for all  $h \in S_q$  and so  $X_1^Q = S_q$ . Thus  $C$  and  $X$  satisfy the conditions of Theorem 1.1. Now let  $G = \Pi_{i=1}^\ell \text{Diag}_q(S_q)$ . By following arguments that are similar to those used in the proof of Proposition 5.2, we deduce that  $\text{soc}(K)$  is a subgroup of  $\text{soc}(G) = \Pi_{i=1}^\ell \text{Diag}_q(A_q)$ . Moreover, for each  $i$ ,  $T_i = \{(1, \dots, x_t, \dots, 1) : x_t \in \text{Diag}_q(A_q)\}$  is a minimal normal subgroup of  $K$ , so  $T_i \leq \text{soc}(K)$ . Consequently  $\text{soc}(K) = \text{soc}(G)$ . It follows that

$$\Delta = \alpha^{\text{soc}(K)} = \{(\alpha(h_1), \dots, \alpha(h_\ell)) : h_i \in A_q\} = \text{Prod}_\ell(C(A_q)),$$

which is a proper subset of  $C$ . Now, if  $\beta = (\alpha(t_1), \dots, \alpha(t_\ell)) \in C$ , then  $t_i \in A_q$  for some  $i$  if and only if  $t_j \in A_q$  for all  $j$ . From this we deduce that

$$C = \bigcup_{t \in \mathcal{T}} \text{Prod}_\ell(C(A_q t)) = \bigcup_{t \in \mathcal{T}} \Delta^{(x_t, \dots, x_t)}$$

where  $\mathcal{T}$  is a transversal for  $A_q$  in  $S_q$ .

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